ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Գասպարյան Արա Գուրգենի

Ուղղությունից կախված լարի երկարության բաշխումը և կովարիոգրամը սահմանափակ ուռուցիկ մարմինների համար

ՍԵՂՄԱԳԻՐ

Ա.01.05-“Հավանականությունների տեսություն և մաթեմատիկական վիճակագրություն”մասնագիտությամբ ֆիզիկամաթեմատիկական գիտությունների թեկնածուի գիտական աստիճանի հայցման ատենախոսության

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YEREVAN STATE UNIVERSITY

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Orientation dependent chord length distribution and covariogram for bounded convex bodies

ABSTRACT

of dissertation submitted for the degree of candidate of phys-math sciences
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Ատենախոսության թեման հաստատվել է Երևանի պետական համալսարանում:

Գիտական ղեկավար՝ ֆիզ-մաթ գիտությունների դոկտոր
Վ. Կ. Օհանյան

Պաշտոնական ընդիմախոսներ՝ ֆիզ-մաթ գիտությունների դոկտոր
Բ. Ս. Նահապետյան

բնական գիտությունների դոկտոր
Վ. Նագել (Յենա, Գերմանիա)

Առաջատար կազմակերպություն՝ Հայ-ռուսական (Սլավոնական) համալսարան

Պաշտպանությունը կկայանա 2016թ. հունիսի 7-ին ժ. 15:00-ին Երևանի պետական համալսարանում գործող ԲՈՀ-ի 050 մասնագիտական խորհրդի նիստում (0025, Երևան, Ալեք Մանուկյան 1):

Ատենախոսությանը կարելի է ծանոթանալ Երևանի պետական համալսարանի գրադարանում:
Ատենախոսությունը առաքված է 2016թ. մայիսի 6-ին:

Մասնագիտական խորհրդի գիտական քարտուղար Տ. Ն. Հարությունյան

The topic of dissertation was approved by the Yerevan State University

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The dissertation is available in the library of Yerevan State University.
The abstract of dissertation was distributed on 6th of May.

Scientific secretary of specialized council T. N. Harutyunyan

General description

Topicality: Reconstruction of a bounded convex body over its cross sections is one of the main tasks of geometric tomography, a term introduced by R. Gardner in [1]. In the mentioned field of mathematics one tries to identify an unknown geometrical object by data from its lower dimensional sections (such as lines or hyperplanes) or projections onto them. Tomography is mainly engaged in the description of the subclasses of cross sections of a body which can reconstruct the body (see [2]).

However, calculation of the geometrical characteristics of the cross sections is often a difficult task. Reconstruction of convex bodies using random sections makes it possible to simplify the calculation, since one can use the technics of mathematical statistics in order to estimate geometrical characteristics of random sections. Quantities characterizing random sections of the body **D** carry some information on **D** and if there is a connection between geometrical characteristics of **D** and probabilistic characteristics of random cross-sections, then by a sample of results of experiments one can estimate the geometric characteristics of the body.

The question of the existence of a bijection between bounded convex bodies and chord length distribution functions of these bodies was made by German mathematician W. Blaschke (see [3]). This question has received a negative answer (see [4]): it is constructed two non-congruent bounded convex 12-gons with the same chord length distribution. One possible way of treating this problem is to consider subclasses of the class of convex bodies for which the chord length distribution provides sufficient information to distinguish between non-congruent members (see [5] and [6]). Gates in [5] showed that triangles and quadrangles can be reconstructed from their chord length distributions. Similarly, a hypothesis about existence of one-to-one correspondence between orientation dependent chord length distributions and bounded convex bodies can be considered. The concept of orientation dependent chord length distribution function is closely connected with the notion of covariogram which has a great importance for investigation of bounded, convex bodies.

The covariogram of a convex body **D** is a function giving the volume of the intersection of the body with its translation. G. Matheron conjectured that in the plane the covariogram determines **D** within the class of all planar convex bodies, up to translations and reflections in a point.

The first partial solution of Matheron's conjecture gave Nagel (see [7]), who proved that every planar convex polygon is determined within all planar convex polygons by its covariogram, up to translations and reflections. In [9] the authors observed how much of the covariogram information is necessary for the uniqueness of the determination and also extended the class of bodies for which the conjecture was confirmed. In 2009 G. Averkov and G. Bianchi showed that every planar convex body is determined within all planar convex bodies by its covariogram, up to translations and reflections (see [10]).

The covariogram problem in the general case ($n \geq 4$) has a negative answer, as in [8] Bianchi constructed two convex bodies, which cannot be obtained from each other through a translation or a reflection, with the same covariogram in n-dimensional Euclidean space ($n \geq 4$). It has also known that every convex polytope in 3 dimensional Euclidean space is determined by its covariogram, up to translations and reflections (see [11]). However, in R^3 the covariogram problem in the general case is still open.

The convexity is essential for such reconstruction problems. The authors of [12] have given an example of two non-congruent and non-convex polygons with the same covariogram.

The covariogram problem appears independently in other contexts. Adler and Pyke (see [14]) asked whether the distribution of the difference $X - Y$ of two independent random points X, Y uniformly distributed over $\mathbf{D}$ determines $\mathbf{D}$ up to a translation or a reflection.

The forms of covariogram for subclasses of convex bodies also allows us to understand new properties of covariogram, what functions can be covariograms of convex bodies, what properties does the covariogram have and what geometric properties can be understood in the covariogram.

It is considered in the dissertation a random segment $L(\omega)$ in R^n with fixed direction and length under the assumption that $L(\omega)$ intersects bounded convex body $\mathbf{D}$. The connection between covariogram and the distribution of the random length of $L(\omega)$ is obtained. A necessary condition for orientation dependent chord length distribution function as a function of maximal chord is obtained. A class of parallelograms for which the necessary condition does not satisfied is constructed. A question of recognition of convex bodies in R^2 from orientation dependent chord length distribution function in finite directions is also considerd.

Objective:

1) Obtaining the explicit form of covariogram for subclass of convex domains.

2) Providing a connection between covariogram and orientation dependent random segment distribution function.

3) Finding a necessary condition for orientation dependent chord length distribution function to be a function of maximal chord.

4) Considering a recognition problem of convex bodies from orientation dependent chord length distribution function in finite directions.

Research methods: Methods from Integral and Stochastic Geometry and Probability Theory.

Scientific novelty: All result presented in the dissertation are new.

Practical and theoretical significance: The main results of the work are of theoretical nature but also have possible practical applications in crystallography.

Approbation: The results are presented in the scientific seminars of Stochastic and Integral Geometry in the Chair of Probability Theory and Mathematical Statistics of Yerevan State University, in the Institute of Mathematics of National Academy of Sciences of Armenia and in the Institute of Mathematics of Friedrich Schiller University, Jena, Germany. Also, results are presented in the conferences "Armenian Mathematical Union, Annual Session dedicated to 1400 anniversary of Anania Shirakatsy, 2012, Yerevan", "Armenian Mathematical Union, Annual Session dedicated to 90 anniversary of Rafael Alexandrian, 2013, Yerevan", "Second international conference Mathematics in Armenia: advances and perspectives, 2013, 24 - 31 August, Tsaghkadzor", "Armenian Mathematical Union Annual Session, 2014, Yerevan", "International science and applied conference: actual problems of mathematical modeling and computer science, 2015, 15–24 May, Sochi" and "Armenian Mathematical Union, 7th Annual Session dedicated to the 100th anniversary of Professor Haik Badalyan, 2015, Yerevan".

Main results of the dissertation are published in five papers; references can be found in the end of this booklet.

Structure and volume of dissertation: the dissertation is written on 77 pages; consists of an introduction, four chapters, conclusion and the list of 45 cited references.

Overview and main results

Let $R^n (n \geq 2)$ be the n-dimensional Euclidean space, $\mathbf{D} \subset R^n$ be a bounded convex body with inner points and V_n be n-dimensional Lebesgue measure in R^n.

Definition. *(see [13]). The function*

$$C(\mathbf{D}, h) = V_n(\mathbf{D} \cap (\mathbf{D} + h)), \quad h \in R^n,$$

is called the covariogram of $\mathbf{D}$*. Here* $\mathbf{D} + h = \{x + h, \ x \in \mathbf{D}\}$*.* $C(\mathbf{D}, h)$ *is also called set covariance of* $\mathbf{D}$*.*

The definition of covariogram is given by G. Matheron, who formulated it for more general sets, and even for functions.

Denote by S^{n-1} the $(n-1)$-dimensional sphere of radius 1 centered at the origin in R^n. We consider a random line which is parallel to $u \in S^{n-1}$ and intersects $\mathbf{D}$, that is an element from

$$\Omega_1(\mathbf{D}, u) = \{\text{line which is parallel to } u \text{ and intersect } \mathbf{D}\}.$$

Let $\Pi r_{u^\perp}\mathbf{D}$ be the orthogonal projection of $\mathbf{D}$ on the hyperplane $u^\perp$ ($u^\perp$ is the hyperplane with normal u and passing through the origin). A random line which is parallel to u and intersects $\mathbf{D}$ has an intersection point (denote by x) with $\Pi r_{u^\perp}\mathbf{D}$. We can identify the points of $\Pi r_{u^\perp}\mathbf{D}$ and the lines which intersect $\mathbf{D}$ and parallel to u. It means, that we can identify $\Omega_1(\mathbf{D}, u)$ and $\Pi r_{u^\perp}\mathbf{D}$. Assuming that the intersection point x is uniformly distributed over the convex body $\Pi r_{u^\perp}\mathbf{D}$ we can define the following function:

Definition. *The function*

$$F(\mathbf{D}, u, t) = \frac{V_{n-1}\{x \in \Pi r_{u^\perp}\mathbf{D} : V_1(g(\mathbf{D}, u, x) \cap \mathbf{D}) < t)\}}{b(\mathbf{D}, u)}$$

is called orientation dependent chord length distribution function of $\mathbf{D}$ *in direction* u *at point* $t \in R^1$*, where* $g(\mathbf{D}, u, x)$ *- is the line which is parallel to* u *and intersects* $\Pi r_{u^\perp}\mathbf{D}$ *at point* x *and* $b(\mathbf{D}, u) = V_{n-1}(\Pi r_{u^\perp}\mathbf{D})$*.*

Now we can provide Matheron's Lemma, where the connection between the covariogram and the orientation dependent chord length distribution function is observed. One can introduce every vector $h \in R^n$ by $h = (u, t)$, where u is the direction of h, and t is the length of h.

Lemma 1. *(see [13]). Let $u \in S^{n-1}$ and $t > 0$ such that $\mathbf{D} \cap (\mathbf{D} + tu)$ contains inner points. Then $C(\mathbf{D}, u, t)$ is differentiable with respect to t and*

$$-\frac{\partial C(\mathbf{D}, u, t)}{\partial t} = (1 - F(\mathbf{D}, u, t)) \cdot b(\mathbf{D}, u). \tag{1}$$

At $t = 0$ the right-hand derivative exists, and the same equation holds.

From (1) it is easy to see, that

$$b(\mathbf{D}, u) = -\frac{\partial C(\mathbf{D}, u, t)}{\partial t}\bigg|_{t=0}.$$

The explicit form of covariogram help us to solve many probabilistic problems (see [29]), like calculation of explicit forms of orientation dependent chord length distribution function. In the Chapters 1 and 2 of the dissertation the explicit formula for covariogram and orientation dependent chord length distribution function is obtained for every triangle and parallelogram.

Let G_n be the space of all lines g in R^n. A line $g \in G_n$ may be determined by its direction $u \in S^{n-1}$ and its intersection point x in the hyperplane $u^{\perp}$. The density $du^{\perp}$ is the volume element du of the unit sphere S^{n-1} and dx is the volume element of $u^{\perp}$ at x. Let $\mu(\cdot)$ be the locally finite measure on G_n, invariant under the group of Euclidian motions. It is well known that the element of $\mu(\cdot)$ up to a constant factor has the following form (see [15], [16])

$$\mu(dg) = du\, dx$$

Denote by O_n the surface area of the n-dimensional unit sphere (note that $O_0 = 2$). For each bounded convex body $\mathbf{D}$, we denote the set of lines that intersect $\mathbf{D}$ by

$$[\mathbf{D}] = \{g \in G_n,\ g \cap \mathbf{D} \neq \emptyset\}$$

We have (see [3])

$$\mu([\mathbf{D}]) = \frac{O_{n-2} V_{n-1}(\partial \mathbf{D})}{2(n-1)},$$

where $\partial \mathbf{D}$ is the boundry of $\mathbf{D}$.
A random line in $[\mathbf{D}]$ is one with distribution proportional to the restriction of μ to $[\mathbf{D}]$.

Definition. *The function*

$$F(\mathbf{D}, t) = \frac{\mu(\{g \in [\mathbf{D}],\ V_1(g \cap \mathbf{D}) < t\})}{\mu([\mathbf{D}])}, \quad \textit{where} \quad t \in R^1$$

is called chord length distribution function of $\mathbf{D}$.

The determination of the chord length distribution function has a long tradition of application to collections of bounded convex bodies forming structures in metals and ceramics. The series of formulae for chord length distribution functions may be of use in finding suitable models when empirical distribution functions are given (see [17]). The results concerning certain infinite cylinders in which the bases of cylinders are a regular triangle or a rectangle are in [18] and [19]. In [20] considered infinite cylinders with regular pentagonal and regular hexagonal bases.

From the definition of the chord length distribution function of $\mathbf{D}$, in order to calculate $F(\mathbf{D},t)$, we have to calculate the following expression:

$$\mu(\mathbf{D},t) = \mu(\{g \in [\mathbf{D}],\ V_1(g \cap \mathbf{D}) < t\})$$

To obtain the explicit formula for the chord length distribution function is an easy task only in the case when $\mathbf{D}$ is a disc. In 1961 Sulanke obtained $F(\mathbf{D},t)$ for a regular triangle [21] and in 1988 $F(\mathbf{D},t)$ is obtained for a rectangle [22] by Gille. The method, which is using δ-formalism in Pleijel identity (see [23]), allowed to calculate the chord length distribution functions for a regular pentagon (see [24]) and regular hexagon (see [25]). In [26] an elementary expression for the chord length distribution function of a regular polygon is given. Moreover, using the inclusion-exclusion principle and the Pleijel identity, an algorithm for calculation of chord length distribution function for a bounded convex polygon is obtained (see [27]). It is also obtained a formula for the chord length distribution function of a lens (see [28]).

In this dissertation a new method is introduced for calculation of $\mu(\mathbf{D},t)$ (see [34] and [35]). This method is based on Matheron's Lemma and integration of the derivative of covariogram:

$$\mu(\mathbf{D},t) = \int_0^{\pi} b(\mathbf{D},u) \cdot F(\mathbf{D},u,t)\, du$$

In Chapter 2 this method is used in order to obtain the explicit forms of chord length distribution functions for every triangle and for every parallelogram. For the chord length distribution function we obtain 3 cases of triangles and 7 cases of parallelograms.

One of the main results in the dissertation is the result from [36], which is providing the connection between the covariogram and orientation dependent distribution of the length of a random segment (see Theorem 3). We consider random segment $L(\omega)$ with length $l > 0$, which is parallel to the fixed direction $u \in S^{n-1}$ and intersects $\mathbf{D}$, and a random variable

$$|L|(\omega) = V_1(L(\omega) \cap \mathbf{D}), \quad \omega \in \Omega_L(\mathbf{D},u)$$

where

$$\Omega_L(\mathbf{D},u) = \{\text{segment with length } l\text{, which is parallel to } u \text{ and intersect } \mathbf{D}\}.$$

Every random segment $L(\omega)$ lies on a line $g(\mathbf{D},u,x)$, therefore $L(\omega)$ can be specified by the coordinates $(g(\mathbf{D},u,x),y)$, where y is the one-dimensional coordinate of the center of $L(\omega)$ on the line $g(\mathbf{D},u,x)$ (as the origin on the line $g(\mathbf{D},u,x)$ we take one of the intersection points of $g(\mathbf{D},u,x)$ and $\mathbf{D}$). Using the mentioned notations we can identify $\Omega_L(\mathbf{D},u)$ with the following set:

$$\Omega_L(\mathbf{D},u) = \left\{(x,y) : x \in \Pi r_{u^\perp}\mathbf{D}, \quad y \in \left[-\frac{l}{2}, \chi(\mathbf{D},u,x) + \frac{l}{2}\right]\right\},$$

where $\chi(\mathbf{D},u,x) = V_1(g(\mathbf{D},u,x) \cap \mathbf{D})$.

$\Omega_L(\mathbf{D},u)$ does not depend on which one of the intersection points of $g(\mathbf{D},u,x) \cap \mathbf{D}$ is taken as the origin. Which of two directions is considered as a positive direction, follows from the explicit form of range of variation of y. Further, we set

$$B_{\mathbf{D}}^{u,t} = \{(x,y) \in \Omega_L(\mathbf{D},u) : \ |L|(x,y) < t\}, \quad t \in R^1,$$

It is obvious, that $\Omega_L(\mathbf{D},u)$ and $B_{\mathbf{D}}^{u,t}$ are measurable subsets of R^n.

Definition. *The function*

$$F_{|L|}(\mathbf{D},u,t)=\frac{V_n(B_{\mathbf{D}}^{u,t})}{V_n(\Omega_L(\mathbf{D},u))}=\frac{1}{V_n(\Omega_L(\mathbf{D},u))}\int_{B_{\mathbf{D}}^{u,t}}dx\,dy$$

is called orientation dependent distribution of the length of a random segment L in direction $u\in S^{n-1}$, where dx is the $(n-1)$-dimensional Lebesgue measure and dy is the 1-dimensional Lebesgue measure.

It is obtained in the Chapter 3 of the dissertation the following five Theorems:

Theorem 1. *Let $u\in S^{n-1}$ and $t\geq 0$. Then*

$$F_{|L|}(\mathbf{D},u,t)=\begin{cases}0 & \text{for} \quad t\leq 0,\\ \dfrac{b(\mathbf{D},u)\left[2t+F(\mathbf{D},u,t)(l-t)-\int_0^t F(\mathbf{D},u,z)dz\right]}{V_n(\mathbf{D})+lb(\mathbf{D},u)} & \text{for} \quad 0\leq t\leq l,\\ 1 & \text{for} \quad t>l.\end{cases}$$

In [30], [34] and [35] the explicit forms of orientation dependent chord length distribution function for triangles, ellipses, regular polygons and parallelograms is given correspondingly, therefore, using Theorem 1 we get elementary expressions for $F_{|L|}(\mathbf{D},u,t)$ for the mentioned planar convex domains.
If we have $F_{|L|}(\mathbf{D},u,t)$ we can reconstruct orientation dependent chord length distribution function $F(\mathbf{D},u,t)$ on the interval $[0,l)$.

Theorem 2. *Let $u\in S^{n-1}$ and $t\in[0,l)$. Then*

$$F(\mathbf{D},u,t)=\frac{V_n(\mathbf{D})+lb(\mathbf{D},u)}{(l-t)b(\mathbf{D},u)}\left[F_{|L|}(\mathbf{D},u,t)+\frac{1}{l-t}\int_0^t F_{|L|}(\mathbf{D},u,z)dz\right]+1-\frac{l^2}{(l-t)^2}$$

The connection between the distribution function of the random variable $|L|(\omega)$ and covariogram over the interval $[0,l]$ is given in Theorem 3:

Theorem 3. *Let $u\in S^{n-1}$ and $t\in[0,l]$. Then*

$$F_{|L|}(\mathbf{D},u,t)=\frac{1}{V_n(\mathbf{D})+lb(\mathbf{D},u)}\left[\frac{\partial C(\mathbf{D},u,t)}{\partial t}(l-t)-C(\mathbf{D},u,t)+V_n(\mathbf{D})+lb(\mathbf{D},u)\right]$$

The values of $F_{|L|}(\mathbf{D},u,t)$ equals 0, when $t\leq 0$ and equals 1, when $t>l$.

If we have $F_{|L|}(\mathbf{D},u,t)$, we can get the values of the derivative of covariogram with respect to t over the interval $[0,l)$.

Theorem 4. *Let $u\in S^{n-1}$ and $t\in[0,l]$. Then*

$$-\frac{\partial C(D,u,t)}{\partial t}=\frac{l^2b(\mathbf{D},u)}{(l-t)^2}-\frac{V_n(\mathbf{D})+lb(\mathbf{D},u)}{l-t}\left[F_{|L|}(\mathbf{D},u,t)+\frac{1}{l-t}\int_0^t F_{|L|}(\mathbf{D},u,z)dz\right]$$

for all t over the interval $[0,l)$.

Finally, it is obtained the connection between the distribution function of the length of random segment within $\mathbf{D}$ and chord length distribution function of $\mathbf{D}$ in R^n:

Theorem 5. *For every bounded convex body* $\mathbb{D}$ *the following equation holds:*

$$F_{|L|}(\mathbf{D},t)=\begin{cases}0 & \textit{for} \quad t\le 0,\\ \dfrac{O_{n-2}V_{n-1}(\partial\mathbf{D})\Big(2t+F(\mathbf{D},t)(l-t)-\int_0^t F(\mathbf{D},z)dz\Big)}{(n-1)O_{n-1}V_n(\mathbf{D})+lO_{n-2}V_{n-1}(\partial\mathbf{D})} & \textit{for} \quad t\in[0,l],\\ 1 & \textit{for} \quad t>l.\end{cases}$$

Under the assumption, that $F(\mathbf{D},t)$ has a density, Theorem 5 is the result from [32].

The relationships between the covariogram and the orientation dependent chord length distribution function of a cylinder and those of its base is established in [31]. Due to these relationships one can find expressions for the covariogram and the orientation dependent chord length distribution function of a cylinder with parallelogramar, elliptical and triangular bases.

An interesting property of orientation dependent chord length distribution function and covariogram is observed in Chapter 4. In some cases $C(\mathbf{D},u,t)$ and $F(\mathbf{D},u,t)$ of convex body $\mathbf{D}$ depends on maximal chord $t_{max}(\mathbf{D},u)$ in direction u.

It is proved in [34] that the covariogram of triangle ΔABC has the following form:

$$C(\Delta ABC,u,t)=\begin{cases}S\left(1-\frac{t}{t_{\max}(\Delta ABC,u)}\right)^2, & t\in[0,t_{\max}(\Delta ABC,u)]\\ 0, & t\ge t_{\max}(\Delta ABC,u)\end{cases}$$

where S is the area of ΔABC.

In [34] also shown that orientation dependent chord length distribution function of ΔABC has uniform distribution for every direction u on the interval $[0,t_{\max}(\Delta ABC,u)]$.

$$F(\Delta ABC,u,t)=\begin{cases}0, & t\le 0,\\ \frac{t}{t_{\max}(\Delta ABC,u)}, & t\in[0,t_{\max}(\Delta ABC,u)],\\ 1, & t\ge t_{\max}(\Delta ABC,u),\end{cases}$$

A natural question arises (see [37]), in which cases are there functions $G_1(x_1,x_2)$ and $G_2(x_1,x_2)$ of two variables such that

$$C(\mathbf{D},u,t)=G_1(t_{max}(\mathbf{D},u),t)$$

and

$$F(\mathbf{D},u,t)=G_2(t_{max}(\mathbf{D},u),t)$$

In Chapter 4, the question formulated above is partially answered, i. e. a necessary condition for orientation dependent chord length distribution function as a function of maximal chord is obtained.

Theorem 6. *If orientation dependent chord length distribution function depends on* u *through* $t_{max}(\mathbf{D},u)$*, then for any two directions* $u_1,u_2\in S^{n-1}$*, such that*

$$t_{max}(\mathbf{D},u_1)=t_{max}(\mathbf{D},u_2)$$

we get

$$b(\mathbf{D},u_1)=b(\mathbf{D},u_2)$$

Chapter 4 considers a question of recognition of convex bodies from orientation dependent chord length distribution function in finite directions, where an example of two non-congruent triangles is constructed with the same orientation dependent chord length distribution functions in finitly many given directions.

Theorem 7. *For any finite subset $A \subset S^1$, there exist two different triangles with the same orientation dependent chord length distribution function for any $u \in A$.*

Theorem 7 provides a counterexample of an open problem formulated in [33].
Let (p, φ) be the polar coordinates of the foot of the perpendicular to $g \in G_2$ from the origin O. Let ϑ be a translation invariant measure on the space G_2. The element of translation invariant measure up to a constant factor has the following form (see [16]):

$$\vartheta(dg) = dp\, m(d\varphi)$$

where dp is the 1-dimensional Lebesgue measure and m is a finite measure on S^1.

Consider two independent and identically distributed lines g_1 and g_2 meeting $\mathbf{D}$. Denote by A the set of all pairs (g_1, g_2) intersecting inside $\mathbf{D}$:

$$A = \{(g_1, g_2) \in [\mathbf{D}] \times [\mathbf{D}] : g_1 \cap g_2 \in \mathbf{D}\}$$

We assume that the lines are distributed by translation invariant measure ϑ. Let $P_{\mathbf{D}}(A)$ be the probability of A, defined by the following way:

$$P_{\mathbf{D}}(A) = \frac{(\vartheta \times \vartheta)(A)}{(\vartheta([\mathbf{D}]))^2}$$

It is obtained those translation invariant measures generating random lines, for which $P_{\mathbf{D}}(A)$ achieves its maximum in the case of disc and rectangle. It is also shown that for every p from the intervral $[0, \frac{1}{2}]$ and for every square there is a measure generating random lines such that $P_{\mathbf{D}}(A) = p$ (see [38]).

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Ամփոփագիր

Ատենախոսության մեջ դիտարկված են ուռուցիկ սահմանափակ մարմինների հավանականային բնութագրիչների ուսումնասիրության որոշ հարցեր կովարիոգրամի միջոցով: Ատենախոսությունում ստացվել են հետևյալ արդյունքները՝

- Հաշվված են կովարիոգրամի և ուղղությունից կախված լարի երկարության բաշխման ֆունկցիայի բացահայտ տեսքերը եռանկյունների և զուգահեռագծերի համար: Նաև հաշվված է լարի երկարության բաշխման ֆունկցիայի բացահայտ տեսքը եռանկյունների և զուգահեռագծերի համար: Լարի երկարության բաշխման ֆունկցիայի տեսանկյունից ստանում ենք երեք դեպք եռանկյան և յոթ դեպք զուգահեռագծի համար:

Դիտարկված է ֆիքսած ուղղություն և երկարություն ունեցող $L(\omega)$ պատահական հատվածը R^n–ում, որը հատում է $\mathbf{D}$ ուռուցկ սահմանափակ մարմինը:

- Ստացված է կապ կովարիոգրամի և $|L|(=$ երկարություն $L \cap \mathbf{D})$ պատահական մեծության բաշխման ֆունկցիայի միջև ցանկացած t–ի համար $[0, l]$–ից.

$$F_{|L|}(\mathbf{D}, u, t) = \frac{1}{V_n(\mathbf{D}) + lb(\mathbf{D}, u)} \left[\frac{\partial C(\mathbf{D}, u, t)}{\partial t}(l-t) - C(\mathbf{D}, u, t) + V_n(\mathbf{D}) + lb(\mathbf{D}, u) \right]$$

- Ստացված է կապ $|L|$–ի բաշխման և ուղղությունից կախված լարի երկարության բաշխման ֆունկցիայի միջև $[0, l)$ հատվածի վրա.

$$F(\mathbf{D}, u, t) = \frac{V_n(\mathbf{D}) + lb(\mathbf{D}, u)}{(l-t)b(\mathbf{D}, u)} \left[F_{|L|}(\mathbf{D}, u, t) + \frac{1}{l-t} \int_0^t F_{|L|}(\mathbf{D}, u, z) dz \right] + 1 - \frac{l^2}{(l-t)^2}$$

- Ստացված է կապ $\mathbf{D}$-ն հատող պատահական հատվածի բաշխման ֆունկցիայի և $\mathbf{D}$-ի լարի երկարության բաշխման ֆունկցիայի միջև R^n-ում:

$$F_{|L|}(\mathbf{D}, t) = \begin{cases} 0 & \text{երբ } t \le 0, \\ \dfrac{O_{n-2}V_{n-1}(\partial \mathbf{D})\left(2t + F(\mathbf{D}, t)(l-t) - \int_0^t F(\mathbf{D}, z) dz\right)}{(n-1)O_{n-1}V_n(\mathbf{D}) + lO_{n-2}V_{n-1}(\partial \mathbf{D})} & \text{երբ } t \in [0, l], \\ 1 & \text{երբ } t > l. \end{cases}$$

- Ապացուցված է (տես [34]), որ եռանկյան կովարիոգրամը և ուղղությունից կախված լարի երկարության բաշխման ֆունկցիան կախված են u ուղղությունից $t_{max}(\mathbf{D}, u)$ մաքսիմալ լարի միջոցով:

- Եթե ուղղությունից կախված լարի երկարության բաշխման ֆունկցիան կախված է ուղղությունից $t_{max}(\mathbf{D}, u)$-ի միջոցով, ապա ցանկացած $u_1, u_2 \in S^{n-1}$ ուղղությունների համար այնպիսին, որ

$$t_{max}(\mathbf{D}, u_1) = t_{max}(\mathbf{D}, u_2)$$

հետևում է

$$b(\mathbf{D}, u_1) = b(\mathbf{D}, u_2) :$$

- Կառուցված է զուգահեռագծերի դաս, որոնց համար չի բավարարվում ուղղությունից կախված լարի երկարության բաշխման ֆունկցիայի՝ ուղղությունից մաքսիմալ լարի միջոցով կախված լինելու համար անհրաժեշտ պայմանը:

- Ստացված է բացասական պատասխան [33]–ում ձևակերպված հետևյալ խնդրի համար. գոյություն ունի արդյո՞ք $A = \{u_1, u_2, ..., u_m\}$ ուղղությունների վերջավոր բազմություն այնպես, որ համապատասխան $F(\mathbf{D}, u_1, t), F(\mathbf{D}, u_2, t), ..., F(\mathbf{D}, u_m, t)$ ուղղությունից կախված լարի երկարության բաշխման ֆունկցիաները միարժեքորեն կվերականգնեն սահմանափակ ուռուցիկ փիրույթը:

  Եթե $A \subset S^1$-ն ուղղությունների վերջավոր ենթաբազմություն է, ապա միշտ կարելի է կառուցել երկու տարբեր եռանկյուն, որոնց համար ուղղությունից կախված լարի երկարության բաշխման ֆունկցիաները համընկնում են բոլոր u-երի համար A-ից:

- Հետազոտված է հարթության մեջ անկախ և միապեսակ բաշխված ուղիղների զույգ, որոնք հատում են $\mathbf{D}$ ուռուցիկ սահմանափակ փիրույթը: Դիտարկված է $\mathbf{D}$-ի ներսում ուղիղների հատվելու $P_{\mathbf{D}}(A)$ հավանականությունը: Շրջանի և ուղղանկյան համար կառուցված են պատահական ուղիղները գեներացնող զուգահեռ տեղափոխությունների խմբի նկատմամբ հաստատադիր չափեր, որոնց համար $P_{\mathbf{D}}(A)$-ն ընդունում է իր մեծագույն արժեքը:

  Ցույց է տրված, որ ցանկացած $p \in [0, \frac{1}{2}]$ համար և ցանկացած քառակուսու համար գոյություն ունի պատահական ուղիղները գեներացնող չափ այնպես, որ $P_{\mathbf{D}}(A) = p$:

Արդյունքները ներկայացված են [34]-[38] աշխատանքներում:

Аннотация

В диссертации рассматриваются некоторые вопросы касающиеся исследования вероятностных характеристик ограниченных выпуклых тел с помощью ковариограммы. Получены следующие результаты:

- Явные выражения ковариограммы и зависящей от ориентации функции распределения длины хорды для любого треугольника и параллелограмма. Также получены явные выражения для функции распределения длины хорды треугольника и параллелограмма. С точки зрения вида функции распределения длины хорды получаем три типа треугольников и семь типов параллелограммов.

Рассмотрен случайный отрезок $L(\omega)$ в R^n с фиксированными направлением и длиной при условии, что $L(\omega)$ пересекает ограниченное выпуклое тело $\mathbf{D}$.

- Связь между ковариограммой и распределением случайной величины $|L|$(= длина $L \cap \mathbf{D}$) для $t \in [0, l]$:

$$F_{|L|}(\mathbf{D}, u, t) = \frac{1}{V_n(\mathbf{D}) + lb(\mathbf{D}, u)} \left[\frac{\partial C(\mathbf{D}, u, t)}{\partial t}(l - t) - C(\mathbf{D}, u, t) + V_n(\mathbf{D}) + lb(\mathbf{D}, u) \right]$$

- Связь между распределением $|L|$ и зависящей от ориентации функцией распределения длины хорды на интервале $[0, l)$.

$$F(\mathbf{D},u,t) = \frac{V_n(\mathbf{D}) + lb(\mathbf{D},u)}{(l-t)b(\mathbf{D},u)}\left[F_{|L|}(\mathbf{D},u,t) + \frac{1}{l-t}\int_0^t F_{|L|}(\mathbf{D},u,z)dz\right] + 1 - \frac{l^2}{(l-t)^2}$$

- Связь между функцией распределения длины случайного отрезка, пересекающего $\mathbf{D}$ и функцией распределения длины хорды $\mathbf{D}$ в R^n:

$$F_{|L|}(\mathbf{D},t) = \begin{cases} 0 & \text{для } t \le 0, \\ \dfrac{O_{n-2}V_{n-1}(\partial\mathbf{D})\left(2t + F(\mathbf{D},t)(l-t) - \int_0^t F(\mathbf{D},z)dz\right)}{(n-1)O_{n-1}V_n(\mathbf{D}) + lO_{n-2}V_{n-1}(\partial\mathbf{D})} & \text{для } t \in [0,l], \\ 1 & \text{для } t > l. \end{cases}$$

В [34] доказано, что ковариограмма и зависящая от ориентации функция распределения длины хорды для треугольника являются функциями максимальной хорды $t_{max}(\mathbf{D},u)$ в направлении u.

- Если зависящая от ориентации функция распределения длины хорды зависит от направления только через $t_{max}(\mathbf{D},u)$, тогда для любых двух направлений $u_1, u_2 \in S^{n-1}$ таких, что

  $$t_{max}(\mathbf{D},u_1) = t_{max}(\mathbf{D},u_2)$$

  следует

  $$b(\mathbf{D},u_1) = b(\mathbf{D},u_2).$$

- Приведен класс параллелограммов, для которых не выполняется необходимое условие того, что зависящая от ориентации функциия распределения длины хорды была бы функцией зависящей от максимальной хорды.

- Получен отрицательный ответ на следующую проблему из [33]: существует ли множество конечных направлений $A = \{u_1, u_2, ..., u_m\}$ такое, что соответствующий набор значений $F(\mathbf{D},u_1,t), F(\mathbf{D},u_2,t), ..., F(\mathbf{D},u_m,t)$ зависящих от ориентации функций распределения длины хорды однозначно определяют ограниченную выпуклую область?

  Если A - конечное подмножество направлений из S^1, то всегда можно построить два различных треугольника, для которых значения зависящих от ориентации функций распределения длины хорды совпадают для всех $u \in A$.

- Рассмотрена вероятность $P_{\mathbf{D}}(A)$ того, что две независимые и одинаково распределенные прямые, пересекающие плоскую выпуклую область $\mathbf{D}$ имеют точку пересечения внутри $\mathbf{D}$. Строятся трансляционно-инвариантные меры, генерирующие случайные прямые, при которых $P_{\mathbf{D}}(A)$ достигает максимума для диска и прямоугольника.

  Показано, что для каждого $p \in [0, \frac{1}{2}]$ и для каждого квадрата существует мера генерирующая случайные прямые такая, что $P_{\mathbf{D}}(A) = p$.

Результаты представлены в работах [34]-[38].