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Gor Artur Abovyan

Quantum control of an atom with multi-component fields

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Chapter 1

INTRODUCTION

The current work is dedicated to investigation and development of quantum methods of control of natural and artificial atoms with different multi-component electromagnetic fields.

Recent technological achievements in quantum optics show that a number of improvements in cavity design initiated the investigations of a single atom’s dynamics via its interaction with the quantized cavity field. These studies have opened a new chapter of atom optics where we not only treat the states of the atom and its motion quantum-mechanically, but have also quantized the cavity field. Particularly, cavity quantum electrodynamics in the microwave domain comes closest to this goal. Here, photons are trapped in a superconducting cavity and are probed by atoms crossing the field one at a time [1–4].

One of the basic processes of atomic optics is the deflection of atomic beams interacting with a standing light wave inside an optical cavity [5–7]. The deflection pattern of atoms from a single mode of a quantized cavity field has been investigated in detail. In the quantized field version when the atom is state selectively deflected by the quantum field the splitting on family of dressed states with definite photon numbers takes place (see, for example, [7]). In particular, it has been shown that deflection of atomic beams by a one-mode cavity field is a sensitive function of the photon statistics. Moreover, the translational variables of the atomic center-of-mass are correlated with the cavity field and the internal atomic variables, thus, the information about
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the position of the atom is recorded in the phase and amplitude of the field [5, 6]. In this way, diffraction and interference effects leading to spatial localization of atoms induced by field measurements have also been analyzed [8, 9]. In contrast to the single-mode cavities, the use of bimodal cavities has been found an important step towards the engineering of atomic states as well as nonclassical photonic states [10–17]. The bimodal cavity also possesses the feature of two nearly degenerate light modes of orthogonal polarization that leads to production of entangled states of cavity modes. Recent experiments have demonstrated maximally-entangled photons in different polarization modes of the same cavity [18, 19]. In the experiment [18] the two modes of a superconducting bimodal cavity were prepared in a maximally entangled state by using circular Rydberg atoms while they pass through the bimodal cavity. The role of the atom-cavity detuning in bimodal cavity experiments was discussed in [20]. Most recently, the entanglement of photons in two physically distinct cavities has been also considered [21–23].

Generally speaking the use of the resonant atom-cavity interaction regime is the simplest way to generate entangled states between atoms and/or the field modes of a cavity. In this way models of two or three-level atoms interacting with one or two cavity modes in different configurations were studied extensively (see, for example, [24–26]). Recently, atomic localization and center-of-mass wave-function measurement via multiple simultaneous dispersive interactions of atoms with different standing-wave fields have been investigated [27] in addition to the well known results for single-mode cavity [7–9].

Thus, to advance the atomic optics field consideration of novel effects resulting from the combination of atomic depletion and photonic entanglement in bimodal QED cavity is important. The experimental investigations in this direction seem to be realizable in connection with above mentioned input. In this way, spatially entangled atomic deflections resulting from interaction of a $V$-level atomic beam with a two-mode quantized cavity field have
been considered \cite{28, 29}. It was demonstrated that the deflection properties of an atom passing through two crossed standing light waves are crucially modified when entangled states of light waves are used instead of independent waves. Depending on the atom-light interaction schemes, various periodic two-dimensional patterns in a subwavelength regime were reported.

In this work (chapter 2 section 2.2) continuing the cited articles \cite{28, 29}, we consider atomic deflection in the other configuration that involves interaction of atoms with two co-propagating intracavity standing waves in the entangled state. The other novelty is that both types of atoms with $V$- and $\Lambda$- level structures are considered. This approach allows us to investigate atomic deflection conditioned by entangled states from the more general point of view, on the framework of the Wigner function, on one side and perform analysis for atomic systems where transition through the Raman resonance is realized, on the other side.

The ability to prepare atomic systems in superposition states is important both in fundamental studies of quantum mechanics as well as for various technological applications including the field of quantum information and quantum lithography. It is evident, that application of strong, near-resonant to atomic transitions laser light may result in the production and probing of coherent superposition of atomic or molecular states. In this way, many experiments have been proposed or realized, particularly with single trapped ion \cite{30, 31}, or microwave cavity quantum electrodynamics \cite{32}, with a single Rydberg atom coupled to a single field mode. Creation of such quantum states has also been realized for molecular systems \cite{33} including large organic molecules \cite{34}, for atomic ensembles \cite{31} and even virus \cite{35}. Preparation of the atoms or molecules in the coherent superposition states may lead to substantial changes in optical properties of a medium composed of the particles. Some of the most spectacular examples are: electromagnetically induced transparency (EIT) with extreme change in the group velocity of laser pulses, even including complete stopping of laser pulses (see, review papers
[36, 37]), enhancement of the efficiency of nonlinear optical processes [38, 39] and writing and storage of optical information in meta-stable quantum states [40–42]. The preparation of quantum coherence has also become of paramount importance for the growing field of quantum information science [43–45]. There exist some techniques for probing the quantum interference based on interaction of an atom in a superposition state with field modes. In this way, a final detection is realized by homodyne measurement of states of light field after its interaction with atomic system as well as by the methods of quantum tomography [46]. Particularly, much work has been focused on applications and developments of the technique of quantum tomography for atomic beams, (see [7] and Refs. therein).

In this this work (chapter 2 section 2.3) we demonstrate that production of atomic superposition states is qualitatively displayed in two-dimension patterns of deflected atoms on a two crossed standing waves. The analysis is done in quantum treatment for three-level atoms in Λ - configuration interacting with two crossed standing light waves.

During the past years, problems of precise spatial quantum measurement of atoms as well as the optical techniques for spatial localization of atoms have been subjects of considerable interest. These problems have been mainly considered in interactions of atoms with various internal structures of energetic levels with standing electromagnetic fields. In this way, it has been demonstrated that the measurement of the state of a standing wave during atom-light interaction may lead to the localization of the position of the deflected atom. Localization of the position of atoms passing through standing light wave by making a quadrature phase measurement on the light field was shown in one-dimensional case [8, 47]. Recently, localization and center-of-mass wave-function measurement of four-level atoms using their multiple simultaneous interactions with different standing-wave fields was also demonstrated [27]. The physical meaning of these effects is that the interaction of the atom with a standing-wave laser field has an interaction
strength that depends on the position of the atomic center-of-mass with respect to the nodes of the standing wave [7, 48, 49]. So, if an observable depends on the interaction strength, the atomic position in the standing wave can be reconstructed from the measurement results. For instance, the phase shift on the standing-wave field due to a nonresonant, dispersive interaction with atoms can be measured in this way.

In this work (chapter 2 section 2.4) we discuss formation of two-dimensional spatial structures of atoms within the optical wavelength due to atomic diffraction on two crossed standing electromagnetic fields in optical cavities. This analysis proposed for $V$-type atomic configuration under dispersive atom-field interactions in quantum regime. Here, we demonstrate that localization procedure for narrow initial position distributions of atoms leads to production of controllable two-dimensional pattern structures with feature spacing smaller than a wavelength of light in a cavity.

The other field of our interest is the artificial atoms realized by superconducting circuits based on Josephson junctions. These are promising candidates for studying fundamental physics and implementing qubits and controllable quantum two-level systems for quantum computing (see, for example, [50–53] for reviews). The simplest Josephson-junction (JJ) qubit consists of a small superconducting island with $n$ excess Cooper-pair charges connected by a tunnel junction with capacitance $C_J$ and Josephson coupling energy $E_J$ to a superconducting electrode and the single-electron charging energy $E_C$. In the case of a qubit only two charge states with $n = 0$ and 1 play a role while all other charge states, having a much higher energy, can be ignored. Thus, a superconducting charge qubit [54] behaves as an artificial two-level atom in a Cooper box, which is well described by two charge states, and the electrostatic energy difference between these states is controlled by the normalized gate charge.
When a qubit is driven by an external periodically time-dependent electromagnetic field, it has given rise to new quantum effects such as Rabi oscillations and coherent control [55–58], which are the bases for quantum operations. In a series of experiments many fundamental effects from quantum optics have been demonstrated [59–65], including a lasing effect with a Josephson-junction charge qubit embedded in a superconducting resonator [61]. Superconducting qubits usually have short coherence time; therefore, to decrease the time for performing gate operations a large-amplitude external field should be applied. The dynamics of a qubit driven by large-amplitude external fields in the case of driving around the region of avoided level crossing has been also studied (see, [66] and [67] for reviews). Most studies of qubit dynamics assume the driving field to be monochromatic or a single cavity mode.

In this work (chapter 3) we have investigated dynamics of a qubit and the phenomenon of Rabi oscillations for an artificial two-level atom interacting with a monochromatic field with time-modulated amplitude. Such an external field can be also presented as a bichromatic field that consists of two components of equal amplitudes which are symmetrically detuned from the qubit resonance frequency. In this case, the modulation frequency is displayed as the difference between frequencies of two spectral components. This approach, involving modulation of the energy splitting of a qubit in complicated form due to interaction with an external bichromatic field, is different from the standard scheme of laser physics in which the bichromatic field leads to dipole transitions between two states of atoms. This approach can be also applied for investigation of a wide variety of interesting phenomena including tunneling dynamics of time-dependently driven nonlinear quantum systems. In addition, this problem offers an ideal testing ground for studying the fundamental interactions between qubits and
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multi-spectral component light. Note, that the scheme of the Josephson-junction qubit considered in this chapter seems to be close to the experimental scheme on the frequency-modulated transmon qubit performed most recently in [68].

The other goal of this work was the application of the method of quasienergies (QEs) and quasienergetic states (QESs) (or the so-called Floquet states) for the qubit in a bichromatic field. Note that, at first, the QESs of the composite system consisting of an atom and time-periodic e.m. field have been considered in [69–72]. These states provide a classical counterpart to well-known atomic-dressed states [73] in which the coupling to the laser is described by a classical field, whereas the coupling to the vacuum must be described in second quantization. However, one may still hope that in the limit of a macroscopically relevant laser field, both approaches lead to the same results. On the other hand, a certain advantage of the classical treatment implied by the Floquet approach lies in the fact that laser pulses can be handled more easily than in a fully quantized approach to the field (see, e.g., [74]). In the Floquet picture the QESs of the composite system are formed in a strong external field, and the radiation processes and spectral lines are described by transitions between them due to the interaction of the composite system with an electromagnetic vacuum or with a weak probe field. In this way, the master equations in the QES basis were obtained in [75, 76] and in the dressed-state basis in [77]. Thus, the method of QESs is a powerful theoretical framework for the study of bound-bound multiphoton transitions driven by periodically time-dependent fields (see, for review [78]).

Applications of QESs and quasienergies to Josephson qubits in a driving field have been done in several papers [67, 79–81], including a review paper on Landau-Zener-Stückelberg interferometry [67], probe spectroscopy of QESs [80], application of the Floquet theory to Cooper pair pumping [79], and observation of the Stark effect and generalized Bloch-Siegert shift in the experiment with a superconducting qubit probed by resonant absorption via
Chapter 1. INTRODUCTION

The experiments on the Rabi oscillations in monochromatically driven Josephson qubits have been performed and interpreted on the basis of dressed states [56, 57].

We believe that the results of forming atomic spectral lines with strongly different frequencies under bichromatic radiation are important also for the superconducting qubit inducing additional Rabi oscillations on quasienergetic states of the qubit. Additionally, we demonstrate below that quasienergetic states and quasienergies of the bichromatically driven superconducting qubit under consideration differ drastically from the analogous well-known states of the standard two-level atom in a bichromatic field, and due to this difference unusual field-dependence effects appear for the qubit.

Note, that time modulation of quantum dynamics for some systems allows effective control of dissipation and decoherence effects, essentially improving the quantum effects. Indeed, it has been shown that the time modulation in an optical parametric oscillator leads to improvement of squeezing and continuous-variable entanglement of generated modes [82, 83], and application of such an approach to an anharmonic oscillator leads to preparation of oscillatory Fock states’ superpositions in the presence of decoherence [84, 85]. Thus, we expect that this approach applied to artificial atoms, particularly superconducting qubits, will lead to obtaining new qualitative quantum effects involving control of superconducting qubits and improvement of decoherence.

In this work (Chapter 3) we have presented analytical results for nontrivial dynamics of a qubit in a time-modulated field (a bichromatic field), particularly, considering in detail time-dependent populations of qubit states. We have calculated QESs and quasienergies of the composite system ”superconducting qubit plus time-modulated field” in resonance approximation by using the Furry picture.

Quantum information technology with the qubits based on trapped ions,
atoms in resonator, quantum wells and superconducting systems is usually realized on the two-level systems under the influence of optical or microwave pulses. However, the general analytical solution of dynamics of this system is not found until now, in particular, in calculations of quantum transitions probabilities between the qubit states without various approximations. The known approximation is the resonance one (approximation of rotating wave: RWA) in which the effects of oscillating terms are neglected [86–88]. Recently the strong coupling regimes of two-level quantum system with the radiation field has been considered, where the effects beyond RWA become important [89, 90]. The investigations of phenomena beyond the framework of RWA are cited in a series of works [91–97]. The resonant interaction of atom with the bichromatic field is considered in [75, 76, 98–103], and in the case of amplitude modulation in [104]. The resonant effects of atom interaction with the laser field are fully investigated in the nonlinear [105, 106] and atom [107] optics.

In the present work (chapter 4 section 4.1) an approach is given which enables one to generalize the investigation of two-level atom system beyond the limits of the resonance approximation. It is known that the investigation of the perturbed two-level system is simplified when the Bloch variables are used for the vector of state [108, 109]. In the approach supposed below, the matrix of temporal evolution of quantum state in the Magnus formulation [110] described in the terms of Bloch variables is used. Within the frames of such an approach one can obtain the general analytical expressions for transition probabilities between the states of an atom beyond the resonant approximation.

Interactions of quantum systems with pulse train are of great interest for many scientific and technological applications, such as qubit manipulation or quantum computing, secure communications and ultra-precise measurements. This direction is similar to well known method of nuclear magnetic resonance for the radio-frequency control and manipulation of spin systems,
and has also been applied in atomic physics (see, for example: [111], [112]), in development of ultrafast quantum gates with trapped ions [113], in magnetometry with solid state quantum sensors [114], molecular spectroscopy [115] and driven quantum tunneling associated with a wide variety of interesting phenomena and effects [116].

On the other hand, decoupling methods based on the multiple pulse techniques has been proposed for decoherence control of open quantum systems [117–119]. The experimental applications have been obtained for trapped ions [120–122], atomic ensembles [123] and spin-based devices [124].

Recently, manipulation and control of the states of single superconducting qubit by resonant microwave pulses have been extensively studied [55, 57, 125–127]. Particularly, the dynamics of superconducting qubit driven by external field with time-modulated amplitude and the phenomenon of Rabi oscillations have been considered [104] close to the experimental scheme on the frequency-modulated transmon qubit [68].

In the current work (chapter 4 section 4.2) we present a systematic approach for investigation of tunneling quantum dynamics of two-level systems, particularly, superconducting qubits interacting with a single pulse as well as with a train of pulses. In this way, we analyse the Rabi model on base of the Magnus QED formalism in Furry representation for calculation of time-evolution operator. The Magnus formalism [110, 128] leads to expansion of the time-evolution operator in the exponential form that contains all terms of perturbation theory on coupling constant without Dyson time-ordering in QED. The advantages of this approach for periodically driven systems recently have been discussed in details [129]. The Furry picture (see, for example [130]) gives a simple tool for describing processes in an external field. Thus, this approach allows us to obtain general explicit solutions for single-qubit populations as a function of time-dependent Rabi frequency and allows to apply RWA as well as to consider naturally the system beyond RWA. Indeed, by using the Magnus formalism in the Furry representation
we can formulate an effective truncation procedure for Magnus series: in RWA the operator of time-evolution is only determined by the first term of Magnus series and effects beyond RWA can be calculated as corrections to the resonant part through the high-order Magnus terms.

In this way, we demonstrate the results for a wide range of frequencies of driving field corresponding to multiphoton resonant excitation regimes of qubit for various pulse regimes. The concrete calculations will be done for the train of pulses with Gaussian envelopes and with different phases. The duration of pulses, time intervals between them and the relative phases of pulses are the free control parameters.

The generalizations of Furry-Magnus approach to the system of two interacting qubits are also presented in this work (chapter 4 section 4.3).

The goals of this work:

- The investigation of atomic beam’s deflection on the quantum mechanically entangled standing waves.
- Development of a procedure for two-dimensional spatial localization of atomic beam conditioned by quadrature phase measurement on the light fields.
- The investigation of possibilities to visualize the internal state of deflected atoms in their position and momentum distributions.
- The analysis of the dynamics of a superconducting qubit and the phenomenon of multiorder Rabi oscillations in the presence of a time-modulated external field.
- The investigation of dynamics of a superconducting artificial atom interacting with a train of pulses.
• The investigation of qubits interacting with the perturbation field of an arbitrary envelope in the Furry representation based on the Magnus expansion in quantum electrodynamics.

**Scientific novelty:**

• Deflection of atomic beams on two co-propagating standing waves excited in a bimodal cavity has been investigated. The novelty is that the standing waves are in an entangled, photon-number correlated state and interactions with three-level atoms of V-type and Λ-type configurations are considered.

• The measurement-induced-localization procedure is applied to both the large-dispersive and Raman-resonant regimes of Λ atom-field interaction scheme, calculating the conditional position distribution of atoms while considering the two-mode field to be in a given reference quadrature-phase state.

• The dynamics of a superconducting qubit and the phenomenon of multiorder Rabi oscillations in the presence of a time-modulated external field have been analyzed.

• The system of qubits interacting with the perturbation field of an arbitrary envelope in the Furry representation based on the Magnus expansion in quantum electrodynamics is investigated.

• Dynamics of a two-level quantum system in an electromagnetic excitation field is investigated in the Magnus representation on the Bloch sphere. This approach leads to the general expressions for the probability of quantum transitions beyond the resonance approximation.
• The manipulations of a qubit by a train of pulses has been investigated in a systematic approach based on the Magnus expansion and Furry representation in quantum electrodynamics.

**Basic thesis statements:**

• The results of investigation of atomic states splitting, quantum interference effects and spatial atomic localization due to phase measurements on the framework of both the conditional position distributions of atomic wave packets as well as the Wigner functions for atomic translation variables have been presented for interaction of entangled co-propagating standing waves with $V$- and $\Lambda$-type of atoms.

• An approach has been developed for the testing and visualization of atomic superposition states as well as for probing the Raman resonance in a new nonspectroscopic manner.

• A technic has been developed for realization of several "spatial filter functions" by field quadrature measurement.

• An analysis of the dynamics of a superconducting qubit and the phenomenon of multiorder Rabi oscillations in the presence of a time-modulated external field have been presented. This approach leads to obtaining qualitative quantum effects beyond those for the case of monochromatic excitation of qubits. The calculations of Floquet states and quasienergies of the composite system superconducting qubit plus time-modulated field for various resonant regimes are presented.

• Dynamics of a two-level quantum system in an electromagnetic excitation field is investigated in the Magnus representation on the Bloch sphere. This approach leads to the general expressions for the probability of quantum transitions beyond the resonance approximation.
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- A systematic approach for investigation of tunneling quantum dynamics of two-level systems, particularly, superconducting qubits interacting with a train of pulses has been developed based on the Magnus QED formalism in Furry representation.

**The practical value:**

The results presented in this work are of great importance both for fundamental studies of quantum mechanics as well as for various technological applications. In particular, the approach for testing and visualization of atomic superposition states and the technic developed for realization of several "spatial filter functions" by field quadrature measurement presented in Chapter 2 can have important applications in the field of quantum information and quantum lithography. Also, the systematic approach developed in Chapters 3 and 4 for investigation of tunneling quantum dynamics of two-level systems interacting with a time-modulated pulse as well as with a train of pulses, can be applied in fields such as qubit manipulation or quantum computing, secure communications and ultra-precise measurements.

**Abbreviations:**

- QE - quasienergy
- QES - quasienergetic state
- QED - quantum electrodynamic
- RWA - rotating wave approximation
- PNCS - photon-number correlated state

**Structure of the thesis:**
Chapter 1. INTRODUCTION

The thesis consists of 5 chapters and bibliography. It presented in 150 pages, including 40 figures and 175 bibliographic references.

The chapter 1 is an introduction to this work.

The chapter 2 is devoted to investigation and development of methods for multi-component field control of natural atoms. The technics for testing and visualization of superposition states of atoms are presented.

The chapter 3 presents analytical results for nontrivial dynamics of a qubit in a time-modulated field (a bichromatic field), particularly, considering in detail time-dependent populations of qubit states. Also, the calculations for QESs and quasienergies of the composite system superconducting qubit plus time-modulated field” in resonance approximation by using the Furry picture are presented.

The chapter 4 presents an investigation for a dynamics of a two-level quantum system in the Magnus representation on the Bloch sphere and a study of qubit manipulations by a train of pulses in a systematic approach based on the Magnus expansion and Furry representation.

The Chapter 5 is the conclusion of the thesis.
Chapter 2
CONTROL OF NATURAL ATOMS WITH MULTI-COMPONENT FIELDS

2.1 Introduction

One of the basic processes of atomic optics is the deflection of atomic beams interacting with a standing light wave inside an optical cavity [5–7]. The deflection pattern of atoms from a single mode of a quantized cavity field has been investigated in detail. In the quantized field version when the atom is state selectively deflected by the quantum field the splitting on family of dressed states with definite photon numbers takes place (see, for example, [7]). In particular, it has been shown that deflection of atomic beams by a one-mode cavity field is a sensitive function of the photon statistics. Moreover, the translational variables of the atomic center of-mass are correlated with the cavity field and the internal atomic variables, thus, the information about the position of the atom is recorded in the phase and amplitude of the field [5,
In this way, diffraction and interference effects leading to spatial localization of atoms induced by field measurements have also been analyzed [8, 9]. In contrast to the single-mode cavities, the use of bimodal cavities has been found an important step towards the engineering of atomic states as well as nonclassical photonic states [10–17]. The bimodal cavity also possesses the feature of two nearly degenerate light modes of orthogonal polarization that leads to production of entangled states of cavity modes. Recent experiments have demonstrated maximally-entangled photons in different polarization modes of the same cavity [18, 19]. In the experiment [18] the two modes of a superconducting bimodal cavity were prepared in a maximally entangled state by using circular Rydberg atoms while they pass through the bimodal cavity. The role of the atom-cavity detuning in bimodal cavity experiments was discussed in [20]. Most recently, the entanglement of photons in two physically distinct cavities has been also considered [21–23].

Generally speaking the use of the resonant atom-cavity interaction regime is the simplest way to generate entangled states between atoms and/or the field modes of a cavity. In this way models of two or three-level atoms interacting with one or two cavity modes in different configurations were studied extensively (see, for example, [24–26]). Recently, atomic localization and center-of-mass wave-function measurement via multiple simultaneous dispersive interactions of atoms with different standing-wave fields have been investigated [27] in addition to the well known results for single-mode cavity [7–9].

Thus, to advance the atomic optics field consideration of novel effects resulting from the combination of atomic depletion and photonic entanglement in bimodal QED cavity is important. The experimental investigations in this direction seem to be realizable in connection with above mentioned input. In this way, spatially entangled atomic deflections resulting from interaction of a $V$-level atomic beam with a two-mode quantized cavity field have been considered [28, 29]. It was demonstrated that the deflection properties
of an atom passing through two crossed standing light waves are crucially modified when entangled states of light waves are used instead of independent waves. Depending on the atom-light interaction schemes, various periodic two-dimensional patterns in a subwavelength regime were reported.

The ability to prepare atomic systems in superposition states is important both in fundamental studies of quantum mechanics as well as for various technological applications including the field of quantum information and quantum lithography. It is evident, that application of strong, near-resonant to atomic transitions laser light may result in the production and probing of coherent superposition of atomic or molecular states. In this way, many experiments have been proposed or realized, particularly with single trapped ion [30, 31], or microwave cavity quantum electrodynamics [32], with a single Rydberg atom coupled to a single field mode. Creation of such quantum states has also been realized for molecular systems [33] including large organic molecules [34], for atomic ensembles [31] and even virus [35]. Preparation of the atoms or molecules in the coherent superposition states may lead to substantial changes in optical properties of a medium composed of the particles. Some of the most spectacular examples are: electromagnetically induced transparency (EIT) with extreme change in the group velocity of laser pulses, even including complete stopping of laser pulses (see, review papers [36, 37]), enhancement of the efficiency of nonlinear optical processes [38, 39] and writing and storage of optical information in meta-stable quantum states [40–42]. The preparation of quantum coherence has also become of paramount importance for the growing field of quantum information science [43–45]. There exist some techniques for probing the quantum interference based on interaction of an atom in a superposition state with field modes. In this way, a final detection is realized by homodyne measurement of states of light field after its interaction with atomic system as well as by the methods of quantum tomography [46]. Particularly, much work has been focused on applications and developments of the technique of quantum tomography for
atomic beams, (see [7]).

### 2.2 Atomic deflection conditioned by entangled light waves

In this section continuing the cited articles [28, 29] we consider atomic deflection in the other configuration that involves interaction of atoms with two co-propagating intracavity standing waves in entangled state. The other novelty is that both types of atoms with \( V \) - and \( \Lambda \) -level structures are considered. This approach allows us to investigate atomic deflection conditioned by entangled states from the more general point of view, on the framework of the Wigner function, on one side and perform analysis for atomic systems where transition through the Raman resonance is realized, on the other side. We assume that atomic wave packet travels the cavity along a direction which is orthogonal to the cavity axis and concentrate on analysis of conditional position distribution of atoms considering the intracavity fields in a given reference phase state that is well known London phase state [5, 7]. This measurement procedure gives rise to phase-dependent atom-photon interactions and leads to localization of atoms, while the proposed scheme is of interest for investigation of nontrivial interference effects and quantum superposition states in atomic deflection pattern. Indeed, we demonstrate that atomic deflection patterns for \( V \)-type configuration are essentially different for the cases of independent two waves or the waves in entangled states. Particularly, strong multi-peak structure of the deflection pattern of atomic wave packet passing through the waves in entangled state is demonstrated. For concrete calculations, as an entangled two-mode state we choose twin-photon states or photon-number correlated states (PNCS). This situation can be probably realized in the above mentioned experiments as well as be realized in bimodal cavities that involve correlation between cavity modes due to parametric down-conversion (see, for example, [131]). Indeed,
down-conversion has been identified early on as a very efficient process for generation of intense, twin-photon states or photon-number correlated states (PNCS). These states have been attracting an increasing interest in recent years for several applications in the field of quantum optics and quantum information technologies. The experiments on quantum correlations among the intensities of twin beams were performed in [132–137]. Twin-photon states, PNCSs have been investigated in the spontaneous type-1 and type-2 parametric down-conversion processes [138–141], while continuous-variable entangled light beams have been generated in optical parametric oscillators (see, for example, [142–146]).

2.2.1 Probability distribution of the atomic position

We consider three-level atoms with \( V \)- and \( \Lambda \)-type of configurations of energy levels passing through a cavity orthogonal to its axis. The cavity involves two co-propagated one-mode standing waves. The longitudinal velocity of atoms considered to be great enough to treat it classically and constant during the interaction of atoms with the field. The atomic flux is adjusted so that only one atom interacts with the standing waves at a time. This situation can be realized, in particular, in the following scheme shown in 2.1. The configurations of three-levels atoms are illustrated in 2.11. Our aim here is to investigate the position patterns of deflected atoms in the \( x \)-direction.

Suppose that initially, i.e. at \( t = 0 \), each atom is prepared in a \( a|1\rangle + b|2\rangle + c|3\rangle \) state and the electromagnetic field is described by two standing quantized modes. Thus, we assume the state vector \( |\Psi(t = 0)\rangle \) of the combined system to be

\[
|\Psi(t = 0)\rangle = \int dx f(x)|x\rangle (a|1\rangle + b|2\rangle + c|3\rangle) \otimes \sum_{n_i, n_j} C(n_i, n_j)|n_i\rangle_1|n_j\rangle_2. \quad (2.1)
\]
Here \( f(x) \) describes at \( t = 0 \) the transverse Gaussian position distribution of the atom as it enters the cavity

\[
|f(x)|^2 = \frac{1}{\sqrt{2\pi}\Delta x} \exp \left[ -\frac{(x - \langle x \rangle)^2}{2(\Delta x)^2} \right],
\]

(2.2)

where \( \Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle} \). \(|n_i\rangle_1\) and \(|n_j\rangle_2\) are the photon-number states of two modes, while \( C(n_i, n_j) \) is the probability amplitude of finding photons of modes.

![Figure 2.1](image1.png)

**Figure 2.1**: Schematic diagram of discussed model. The three level atoms passes through a cavity interacting with two electromagnetic modes.

![Figure 2.2](image2.png)

**Figure 2.2**: The configurations of three energy levels of atom interacting with two cavity modes with coupling constants \( g_1 \) and \( g_2 \) for: (a) \( V \)-type of atoms interacting with modes at equal frequencies and opposite circular polarizations, (b) \( \Lambda \)-type of atoms, (c) \( \Lambda \)-type of atoms with equal energies of sublevels interacting with modes of opposite circular polarizations.

Applying the time evolution operator under the effective Hamiltonian of three-level atom (see sections 2.2.2 and 2.2.3)
\[ |\Psi(t)\rangle = e^{-iH_{\text{eff}}t} |\Psi(0)\rangle, \tag{2.3} \]

then yields

\[ |\Psi(t)\rangle = \int dx \sum_{n_i,n_j} \sum_{m=1}^3 \Phi_{n_i,n_j,m}(x,t)|n_i\rangle_1|n_j\rangle_2|m\rangle|x\rangle. \tag{2.4} \]

Below, we consider the position distribution \( |\langle x|\Psi(t)\rangle|^2 \) by making use additionally the correlation between atomic and field variables in amplitudes \( \Phi_{n_i,n_j,m}(x,t) \). In this way, we focus on a joint measurement strategy calculating the conditional probability amplitude \( \langle \Psi_R|\langle x|\Psi(t)\rangle \) of finding the atom at the position \( x \) provided that two-mode field is in a given reference state \( |\Psi_R\rangle \). Two scheme of the joint measurements have been proposed up to now. In one of these a quadrature measurement on the cavity field has been performed \([8, 9]\). For the case of two-modes this procedure has been discussed in \([27]\). In the other scheme proposed in \([7]\) and used in this paper, the phase states are selected as the reference state

\[ |\Psi_R\rangle = |\varphi_1\rangle_1|\varphi_2\rangle_2, \tag{2.5} \]

where

\[ |\varphi\rangle_i = \frac{1}{\sqrt{2\pi}} \sum_n e^{in\varphi} |n\rangle_i. \tag{2.6} \]

In this case, for the joint position distribution \( W(x, \varphi_1, \varphi_2) \) to find the atom in the state \( |m\rangle \) at position \( x \) and the standing waves with the definite phases \( \{\varphi_1, \varphi_2\} \), the following result is obtained

\[ W_{(1\rightarrow m)}(x, \varphi_1, \varphi_2) = \frac{|f(x)|^2}{(2\pi)^2} \left| \sum_{n_i,n_j} \Phi_{n_i,n_j,m}(x,t)e^{-i(n_i\varphi_1+n_j\varphi_2)} \right|^2. \tag{2.7} \]

Note, that such measurement procedure may localized the position of an
atom at much less scale than the wavelength of the standing waves as this has been demonstrated by making a quadrature phase measurement as well as a phase measurement on light fields. In what follows, the position distribution will be obtained for two cases of quantized modes: (a) the case of statistically independent modes for which
\[ C(n_i, n_j) = C_1(n_i)C_2(n_j); \]
(b) the case of entangled modes
\[ C(n_i, n_j) = A(n_i)\delta_{n_i,n_j}. \]
In the last case the state vector of the intracavity field reads as
\[ |\text{field}\rangle = \sum_n A(n)|n\rangle_1|n\rangle_2, \tag{2.8} \]
where \( A_n \) is an amplitude of photon-pair distribution.

### 2.2.2 V-system

In this section we discuss the interaction of \( V \)-type atoms (2.11(a)) with two electromagnetic modes of cavity at the same frequencies but different, opposite cyclic vector of polarization. The couplings of atom with two modes are determined by the spatial mode functions
\[ \Omega_1(x) = g_1 \sin(kx) \quad \text{and} \quad \Omega_2(x) = g_2 \sin(kx + \theta), \]
where \( k = \omega/c \). \( g_1 = \epsilon_1 \vec{e}_1 \langle 1|\vec{d}|2 \rangle \) and \( g_2 = \epsilon_2 \vec{e}_2 \langle 1|\vec{d}|3 \rangle \) are coupling constants, \( \epsilon_i = \sqrt{\omega_i/2v} \) is the electric field per photon. \( \vec{e}_1 \) and \( \vec{e}_2 \) are the polarization vectors while \( \langle 1|\vec{d}|2 \rangle \) and \( \langle 1|\vec{d}|3 \rangle \) are the dipole moments of the corresponding atomic transitions. \( \Delta_1 = \omega_{21} - \omega \) and \( \Delta_2 = \omega_{31} - \omega \) are the detunings of the atomic transition frequencies \( \omega_i \) from the cavity frequency \( \omega \). We have considered the limiting case of a large atom-cavity detuning, i.e. \( \Delta_i \gg g_i^2 \langle n_i \rangle \), \((i = 1, 2)\), with \( \langle n_i \rangle \) being the mean photon number of the \( i \)th cavity mode. The atoms before interacting with the intracavity field is in ground \( |1\rangle \) state. This interaction is described by the following effective Hamiltonian
\[ H_{\text{eff}} = \sum_{i=1,2} \frac{\Omega_i^2(x)}{\Delta_i} a_i^\dagger a_i \sigma_{11}, \tag{2.9} \]
when $a_i^\dagger$, $a_i$ are the Bosonic creation-annihilation operators for the cavity modes $(i = 1, 2)$, and $\sigma_{11} = |1\rangle\langle 1|$. Because of the large detunings the internal state of atom remains in the initial ground state $|1\rangle$ during the interaction, so in this section we discuss only $W_{(1\rightarrow 1)}$ distribution. The amplitude $\Phi_{n_i,n_j,1}(x,t)$ at time $t$ can be obtained from 2.1, 2.3, 2.4 and 2.9 as

$$\Phi_{n_i,n_j,1}(x,t) = f(x)C(n_i,n_j) \times \exp \left[ i(\chi_1 n_i \sin^2(kx) + \chi_2 n_j \sin^2(kx + \theta)) \right],$$

(2.10)

where $\chi_i = g_i^2 t/\Delta_i$.

The conditional position distribution of scattered atoms in the case of independent modes is calculated as

$$W(x, \varphi_1, \varphi_2) = \frac{|f(x)|^2}{(2\pi)^2} W^{(1)}(x, \varphi_1) W^{(2)}(x, \varphi_2),$$

(2.11)

where

$$W^{(1)}(x, \varphi_1) = \left| \sum_n C_1(n) \exp \left[ in(\chi_1 \sin^2(kx) - \varphi_1) \right] \right|^2,$$

(2.12)

$$W^{(2)}(x, \varphi_2) = \left| \sum_n C_2(n) \exp \left[ in(\chi_2 \sin^2(kx + \theta) - \varphi_2) \right] \right|^2.$$  

(2.13)

Each of the distributions $W^{(i)}(x, \varphi_i)$ describes the interaction of atom with only one electromagnetic mode of cavity in absence of the other. It illustrates the independence of interactions between atom of $V$-type and two independent intracavity modes in dispersive regime. The correlation between electromagnetic modes strongly modified the conditional position distribution 2.11 leading to the following result

$$W(x, \varphi_1, \varphi_2) = \frac{|f(x)|^2}{(2\pi)^2} \left| \sum_n A(n) \exp \left[ in(\chi(x) - \varphi) \right] \right|^2,$$

(2.14)
where \( \varphi = \varphi_1 + \varphi_2 \) and \( \chi(x) = \chi_1 \sin^2(kx) + \chi_2 \sin^2(kx + \theta) \). For concrete calculations we choose \( C_1(n) = C_2(n) = C(n) \) with Poissonian distributions

\[
C(n) = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}, \quad \alpha = \sqrt{N}e^{i\phi}, \tag{2.15}
\]

where \( N = |\alpha|^2 \) is the mean photon number in each of the waves, that means that independent waves are in coherent states. We analyze the case of correlated modes assuming for an analogy that amplitude of photon-pair distribution is also Poissonian

\[
A(n) = e^{-|\xi|^2/2} \frac{\xi^n}{\sqrt{n!}}, \quad \xi = \sqrt{N_c}e^{i\gamma}, \tag{2.16}
\]

where \( N_c = |\xi|^2 \) is the average number of photon pairs. We shortly discuss this point. Note, that photon-pair distribution has been theoretically and experimentally investigated mainly for different regimes of parametric down-conversion. In dependence on the modal structures of twin-photon beam these distributions might be in form from thermal to the Poissonian \([147]\). A proper state for our model seems to be the pair-coherent state with an equal photon number in both the modes \([148]\). This state is determined as \( a_1a_2|\xi\rangle = \xi|\xi\rangle \) and is calculated in the following form \( |\xi\rangle = N_0 \sum_n \xi^n/n!|n\rangle_1|n\rangle_2 \), \( N_0 = 1/\sqrt{I_0(2|\xi|)} \). For the case of small pair-photon numbers expansion coefficient only slightly differs from the Poissonian one.

Obviously, the distributions 2.11, 2.12 and 2.14 crucially depend from the phases \( \varphi_1 \) and \( \varphi_2 \) which concretize the ranges of atom-waves interactions. We now discuss mainly the position patterns due to twin-photon number modes, when distribution 2.14 depends from the sum of the phases, \( \varphi = \varphi_1 + \varphi_2 \). For the real coefficients \( A(n) \) (or \( C(n) \) for the case of independent waves) we choose \( \varphi = 0 \), which gives the best spatial localization of scattered atoms. Thus, the probability \( W(x, \varphi = 0) = W(x) \) describes the selected scattering events only those atoms that have passed the nodes of the field with spatial
mode function $\sin(kx)$. Nevertheless, the conditional distribution depends on a relative phase $\theta$ of two modes. As it was shown, such joint measurement procedure plays the role of a spatial filter. Experimentally, this procedure can be realized by a mechanical slit or mask placed in front of a node of the electromagnetic field. For detailed discussion of this point, see [7]. Thus, according to the formula 2.14 the peaks of the distribution can be calculated from the condition of the maximal interference

$$n[\chi_1 \sin^2(kx) + \chi_2 \sin^2(kx + \theta)] = 2\pi m,$$

(2.17)

where $m \in \mathbb{Z}$.

This approach is well adopted to the strong-coupling regime where the coupling constant of atom-field interaction $g$ is greater than the decay rates of the atomic dipole $\gamma_1$ and the cavity field $\gamma_2$, $g \gg \max(\gamma_1, \gamma_2)$. In this regime and for the atom transit time across the cavity $g^{-1} \ll \Delta t \ll \max(\gamma_1^{-1}, \gamma_2^{-1})$ the atomic relaxation is negligible. So far, several experiments in cavity QED have investigated the interaction of an atom with the field in the strong-coupling regime, $g/\gamma_1 \gg 1, g/\gamma_2 \gg 1$, [1–4]. Note, a typical optical cavity [149] has a lifetime $\Delta t = w/v$, (where $w$-mode wait size and $v$-velocity of atoms) of the order of 40ns while the coupling strength is approximately $200MHz$. These estimations lead to the dimensionless coupling $gt \simeq 8$.

The distributions for both cases with independent and correlated modes are shown in figures 2.3 and 2.4 for particular choice of the parameters, i.e. $\theta = 0$ and $\theta = \pi/2$, for the width of a normalized Gaussian smaller than the wavelength $\lambda$ of the modes, $\Delta x = 0.1\lambda$, the dimensionless coupling parameters $|\chi_1| = |\chi_2| = 10$ and opposite detunings $\Delta_1 = -\Delta_2$. The results for the case of the zero relative phase, $\theta = 0$, are presented in 2.3. The dashed curve describing conditional distribution of atoms for the configuration with independent modes has one central peak and two additional smaller peaks at both sides of the central one. In contrast to that the solid curve corresponding to the case of correlated modes behaves as initial atomic distribution. In this
case the position phase function $\chi(x) = 0$ due to a compensation of modes’ influences on atom leading to vanishing of the splitting.

**Figure 2.3:** The conditional position distribution $W(x)$ of $V$ atoms versus the parameter $kx/\pi$ for $|\chi_1| = |\chi_2| = 10$, $N = N_c = 1$, $\Delta x = 0.1\lambda$, $\Delta_1 = -\Delta_2$, $\theta = 0$. Dashed curve describes the case of independent modes, solid curve describes the case of correlated modes.

**Figure 2.4:** The conditional position distribution $W(x)$ of $V$ atoms versus the parameter $kx/\pi$ for $|\chi_1| = |\chi_2| = 10$, $N = N_c = 1$, $\Delta x = 0.1\lambda$, $\Delta_1 = -\Delta_2$, $\theta = \pi/2$. Dashed curve describes the case of independent modes, solid curve describes the case of correlated modes.

For the other particular value of the relative phase, $\theta = \pi/2$, the distributions are shown in Fig.2.4. Here the differences between distributions describing both cases concern to the vanishing of the central peak in the distribution corresponding to correlated modes as well as to appearing of the new number of peaks as the solutions of equation (2.17).
Note, the presented results on atomic distributions describe various one-dimensional patterns at the exit of the cavity, in so-called near zone of atomic diffraction. Obviously, these patterns are transformed being considered in far diffraction zone. In the end of this section, we shortly discuss this point. Outside the cavity the dynamics of the atom is governed by the free Hamiltonian, \( H_{\text{free}} = \frac{P^2}{2M} \). If we assume that \( t_0 \) is the cavity passing time for atoms then the state vector of the combined system at time \( t + t_0 \) is written as

\[
|\Psi(t + t_0)\rangle = e^{-iH_{\text{free}}t} |\Psi(t_0)\rangle.
\]

(2.18)

Using the expression (2.10) it is easy to rewrite the vector state (2.18) as

\[
|\Psi(t_0 + t)\rangle = \sum_{n_i, n_j} C(n_i, n_j) \int dx S_{n_i, n_j}(x, t) |1, n_i, n_j\rangle,
\]

(2.19)

where the amplitude \( S_{n_i, n_j} \) reads

\[
S_{n_i, n_j}(x, t) = \left( \frac{M}{2\pi t} \right)^{1/2} \int dx' f(x') \exp \left[ iM \left( \frac{x'}{2t} - x \right)^2 \right] \\
\times \exp \left[ i(n_i\chi_1 \sin^2(kx') + n_j\chi_2 \sin^2(kx' + \theta)) \right].
\]

(2.20)
The conditional position distribution describing the deflection of atoms on $z = vt$ length outside the cavity

$$W^F_{(1 \rightarrow 1)}(x, t) = |\langle \varphi_1, \varphi_2 | \langle 1 | \langle x | \Psi(t_0 + t) \rangle \rangle|^2,$$  \hspace{1cm} (2.21)

is calculated according to this relation in the following form

$$W^F_{(1 \rightarrow 1)}(x, t) = \left| \sum_{n_i, n_j} C(n_i, n_j) e^{-i(n_i \varphi_1 + n_j \varphi_2)} S_{n_i, n_j}(x, t) \right|^2.$$  \hspace{1cm} (2.22)

As an application of this general result let us consider the modification of the pattern shown in Fig.2.4 for the case of correlated modes. At the distance $z \approx 50cm$ from the exit of the cavity, the result is shown on the Fig.2.5. As we see, an additional interference is taking place on the distance pattern due to overlapping of neighboring peaks that deforms the pattern.

2.2.3 Λ-system

The other example of our interest is Λ configuration of three level atom interacting with two modes at $\omega_1$ and $\omega_2$ frequencies (Fig.2.11(b)). The energy levels satisfy the following condition $\omega_{31} - \omega_1 = \omega_{32} - \omega_2 = \Delta$ and $\Delta \gg \omega_{21}$. Under such conditions the two-photon transitions between $|1\rangle$ and $|2\rangle$ states are realized effectively with the following effective Hamiltonian of this system

$$H_{eff} = \sum_{i=1,2} \frac{\hbar \Omega_i(x)}{\Delta} a_i^\dagger a_i \sigma_{ii} + \frac{\hbar \Omega_1(x) \Omega_2(x)}{\Delta} (\sigma_{21} a_1^\dagger a_2^\dagger + \sigma_{12} a_1^\dagger a_2),$$  \hspace{1cm} (2.23)

where $\Omega_1(x) = g_1 \sin(k_1 x)$ and $\Omega_2(x) = g_2 \sin(k_2 x + \theta)$ are spatially-dependent Rabi frequencies, $\sigma_{21} = |2\rangle \langle 1|$ and $\sigma_{12} = |1\rangle \langle 2|$ are the raising-lowering operators, and $\sigma_{ii}$ describes the populations of the state $|i\rangle$, $(i = 1, 2)$. 
These expressions are presented for the \(\Lambda\)-type atomic configuration that interacts with two waves at different frequencies \(\omega_1\) and \(\omega_2\) (see, Fig. 2.11(b)). For the case of \(k_1 = k_2\) the equations (2.23) describes the \(\Lambda\)-system (see, Fig. 2.11(c)) that interacts with two waves with equal frequencies and opposite circular polarizations. We suppose the general case, assuming that atoms enter the standing waves in a coherent superposition of two lower states \(a|1\rangle + b|2\rangle\). From (2.1) and (2.23) it is easy to derive the state vector of the system. According to that the amplitudes \(\Phi_{n_i,n_j,m}\) are calculated as

\[
\begin{align*}
\Phi_{n_i,n_j,1}(x, t) &= -A_1 + B_1 \exp \left[ -i \frac{\Omega_1^2 n_i \Omega_2^2 (n_j + 1)}{\Delta} t \right], \\
\Phi_{n_i,n_j,2}(x, t) &= -A_2 + B_2 \exp \left[ -i \frac{\Omega_2^2 (n_i + 1) + \Omega_2^2 n_j}{\Delta} t \right],
\end{align*}
\] (2.24)

where the coefficients are equal to

\[
\begin{align*}
A_1 &= \frac{f(x)}{\Omega_1^2 n_i + \Omega_2^2 (n_j + 1)} \\
&\times \left[ \Omega_1 \Omega_2 \sqrt{n_i(n_j + 1)} C(n_i - 1, n_j + 1) \cdot b - \Omega_2^2 (n_j + 1) C(n_i, n_j) \cdot a \right], \\
B_1 &= \frac{f(x)}{\Omega_1^2 n_i + \Omega_2^2 (n_j + 1)} \\
&\times \left[ \Omega_2^2 n_i C(n_i, n_j) \cdot a + \Omega_1 \Omega_2 \sqrt{n_i(n_j + 1)} C(n_i - 1, n_j + 1) \cdot b \right], \\
A_2 &= \frac{f(x)}{\Omega_1^2 (n_i + 1) + \Omega_2^2 n_j} \\
&\times \left[ \Omega_1 \Omega_2 \sqrt{(n_i + 1)n_j} C(n_i + 1, n_j - 1) \cdot a - \Omega_1^2 (n_i + 1) C(n_i, n_j) \cdot b \right], \\
B_2 &= \frac{f(x)}{\Omega_1^2 (n_i + 1) + \Omega_2^2 n_j} \\
&\times \left[ \Omega_2^2 n_j C(n_i, n_j) \cdot b + \Omega_1 \Omega_2 \sqrt{(n_i + 1)n_j} C(n_i + 1, n_j - 1) \cdot a \right].
\] (2.25)

Considering deflection of \(\Lambda\) atoms we apply the scheme of measurement
discussed in Sec.2.2.2. It corresponds to a join measurement of the position of the atom and the phases \(\varphi_1 = 0, \varphi_2 = 0\). This measurement procedure selects only atoms that have not altered the phase of the initial mode \{1\} with mode function \(\sin(k_1x)\) for arbitrary relative phase \(\theta\) of the other mode \{2\}.

According to the restriction \(\Delta \gg \omega_{31}\) in our model the wavevectors \(k_1\) and \(k_2\) satisfy to the condition \(|k_1 - k_2|/k_1 \ll 1\). Correspondingly, the terms depending on variation of wavevectors under such condition do not change the distributions considerably, leading only to small replacement of the peaks. Thus, below we represent the distributions only for the case of equal wavevectors, i.e. for \(k_1 = k_2\). We should also mention that the last term of the Hamiltonian (2.23) describing coherent Raman process involves the product of two Rabi frequencies. In the other words, the Raman process leaves to correlations between two channels of transitions between excited \(|3\rangle\) and two lower states \(|1\rangle\) and \(|2\rangle\) of atoms. Investigating this effect on atomic deflection we note that analogous correlation between transition channels is realized due to interaction of the atoms with two cavity modes in PNCS. It means that for \(\Lambda\)-atoms the large difference between atomic deflections patterns on independent- or entangled-modes is expected to fail in contrast to analogous deflection of \(V\)-atom. For \(V\) configuration two channels of atomic excitations are independent (see, the Hamiltonian (2.9)) and correlation between them is realized only due to interaction of atoms with entangled waves. This situation is demonstrated on figures 2.6, 2.7. We also consider here, an important component of our scheme, the probing of superposition states in deflection of \(\Lambda\)-atoms. The distributions for atoms with initial \(|2\rangle\) and final \(|1\rangle\) states and for the configuration (c)(Fig.2.11) are shown in Fig.2.6. The results for atoms prepared initially in the superposition state \(\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\) are shown in Fig.2.7.

As we see, the deflection pattern of atoms in the lower state is crucially modified when atoms in the superposition state enter a cavity. We conclude that the position distribution of the deflected atom in bimodal cavities allow
us to probe atomic superposition states in a new not spectroscopic approach.

2.2.4 Wigner function

Another tool for investigation of atomic spatial distributions involving effects of quantum interference seems to be the Wigner function $W(x, p)$ in phase space. Indeed, recently this quasi-probability distribution in phase-space has been proposed for investigations of atomic optics in quantized light field (see, for example, [5]). We calculate the Wigner function for atoms passed the bimodal cavity assuming the field of both modes in the given reference $|\Psi_R\rangle$ state. The expression for the Wigner function is given by

$$W(x, p) \equiv \frac{1}{2\pi} \int d\xi \exp(-ip\xi) \left| x + \frac{1}{2}\xi \right| \hat{\rho}_a \left| x - \frac{1}{2}\xi \right\rangle ,$$  (2.26)

where $x$ and $p$ are correspondingly the coordinate and momentum of atom, $\hat{\rho}_a$ is the density operator of an atom

$$\hat{\rho}_a = \langle \Psi_R | \Psi(t) \rangle \langle \Psi(t) | \Psi_R \rangle .$$  (2.27)
On the whole, the Wigner function is expressed through the amplitudes $\Phi_{n_i,n_j,m}(x, t)$ on the formula (2.10) for $V$-system or the formulas (2.24), (2.25) for $\Lambda$-system. The result reads as

$$W(x, p) = \frac{1}{(2\pi)^2} \sum_{m=1}^{3} \sum_{i,j,k,l=0}^{\infty} W_{i,j,k,l;m}(x, p),$$

(2.28)

where

$$W_{i,j,k,l;m}(x, p) = \frac{1}{2\pi} \int d\xi \exp(-ip\xi) \Phi^{*}_{i,j,m}(x + \frac{1}{2}\xi, t) \Phi_{k,l,m}(x - \frac{1}{2}\xi, t).$$

(2.29)

The last expression has derived assuming that $\varphi_1 = \varphi_2 = 0$, i.e. the condition of the best localization.

Below, we restrict ourself by calculation of the Wigner functions for $V$-atom considering two cases presented on the figures 2.3 and 2.4. The Wigner functions for the both cases of the interaction of atoms with independent or entangled modes are shown in Fig.2.8 for the group of parameters used in the Fig.2.3. During the measurement of the reference phase the position distributions (2.11), (2.12) are given by a linear superposition of the dressed states.
amplitudes. That gives rise to the interference pattern in phase-space and negative contributions in the Wigner function. The second pattern (Fig. 2.8(b)) corresponding to the correlated modes by its Gaussian form illustrates the fact that influences of two modes on atom neutralize each other for the field in PNCS.

The Wigner functions displayed in the figures 2.9(a),(b) corresponding to the case of the spatial distributions shown in Fig. 2.4. As we see, there are great difference between both distributions in phase space for two groups of the parameters. The distribution corresponding to correlated modes has incommensurably more maximums and negative parts caused by quantum interference.

In the end of this section we shortly discuss the momentum distribution \( W(p) \) in the deflected pattern of the Gaussian atomic wave packets centered at the position of the best localization (see, section 2.3.2). The results are obtained from the Wigner function by integration over the position variable

\[
W(p) = \int dx W(x, p).
\]
Figure 2.9: Wigner functions for V-atoms in the case of independent modes(a) and correlated modes(b) for the parameters as used in the Fig.2.4, with $\theta = \pi/2$.

We illustrate below the momentum distribution for narrow initial atomic wave packets (with the width $\Delta x = 0.1\lambda$) which obviously correspond to wider distributions in the momentum space. In Fig.2.10 the atomic momentum distribution is shown. Naturally, the conditional momentum distribution should posses some features of the Wigner function. So, in the case of correlated modes and for $\theta = 0$ it consists of only one peak and demonstrates better localization than corresponding distribution for independent waves, which could be anticipated from the position distribution. On the other hand distribution shown in Fig.2.10 (b) for the case of correlated modes and for $\theta = \pi/2$ is spread, two small peaks at both sides of the central main peak are moved further than it is in the second case.
2.3 Visualization of superposition states and Raman processes with two-dimensional atomic deflection

In this section we demonstrate that production of atomic superposition states is qualitatively displayed in two-dimension patterns of deflected atoms on a two crossed standing waves. The analysis is done in quantum treatment for three-level atoms in \( \Lambda \) - configuration interacting with two crossed standing light waves. We demonstrate that the deflection patterns of the atomic beam passing through two crossed standing light waves are modified if the atoms are initially prepared in coherent superposition of lower levels states or as well as when the superposition states are created during the process of deflection. Our other goal is to understand how Raman processes under the general two-photon resonance condition are exhibited in the atomic deflection patterns. Considering different interaction regimes of \( \Lambda \) - system with off-resonant standing waves we have demonstrated that the deflection patterns, in the transverse plane to the direction of the center of atomic mass motion, are essentially different for the case of one-photon and two-photon Raman interactions.
The approach proposed relies on the problem of atomic spatial localization. In general, the precise spatial measurement and localization of quantum particles has been a subject of considerable interest since the discussion on Heisenberg’s famous microscope. A particular class of quantum optical localization schemes suitable to determine the position of a quantum particle on a subwavelength scale makes use of standing wave driving fields (see, [8, 9, 48, 150–155]).

It is well known that when atoms pass a standing-wave cavity mode the strength of interaction with field depends on the positions of the atoms. Thus, quadrature phase measurement on the field leads to strong localization of the atomic position below a wavelength of the field in the cavity [8, 9]. Recently, atomic localization and centre-of-mass wave function measurement via multiple simultaneous dispersive interactions of atoms with different standing-wave fields have been investigated [27] in addition to the well-known results for a single-mode cavity [8, 9, 48, 150–155]. Note, that analogous scheme of atomic deflection has been recently considered for investigation of spatial entanglement in deflection of V-type atoms [28, 156] as well as Λ-type atoms [157]. Thus, we apply the measurement-induced localization procedure for our scheme, calculating the conditional position distribution of atoms while considering two-mode field to be in a given reference quadrature-phase state. As it will be shown here, for narrow initial position distributions of atoms our scheme permits producing controllable pattern structures with feature spacing smaller than a wavelength of the light in the cavity.

For completeness, we also discuss the visualization of atomic superposition states in the momentum space. In this direction the distribution of deflected atoms in terms of the transverse atomic moment is calculated. It is evident that for narrow initial atomic wave packets wider distributions in the momentum space will be realized.
2.3.1 Atomic deflection in the presence of two-photon Raman processes

Let us consider the quantum dynamics of a three-level atom with a $\Lambda$-type configuration of energy levels moving along the $z$-direction and passing through cavities that involve two crossed one-mode standing waves (see, Fig. 2.11). Atomic beam is adjusted so that only one atom interacts with the cavity electromagnetic field at a time and position patterns of deflected atoms in the $x − y$ plane are measured. The transition between the two lower levels $|1\rangle$ and $|2\rangle$ is dipole forbidden and the transition from the upper level $|3\rangle$ to any of the lower $|1\rangle$ and $|2\rangle$ levels is allowed. We focus more specifically on the dispersive limit where the detuning between the two standing wave frequencies and the corresponding atomic transition frequencies are large compared with the Rabi frequencies. We also neglect the atomic damping during the time an atom interacts with the fields.

![Diagram](image)

**Figure 2.11**: Schematic diagrams showing the investigated model. (a) The atomic beam crossing the interaction region. (b) Energetic levels of a $\Lambda$-type of atoms with equal energies of sublevels interacting with modes of opposite circular polarizations with coupling constants $g_1$ and $g_2$.

We consider two standard types of Hamiltonian depending on two detunings $\Delta_1$ and $\Delta_2$. If the frequencies of modes and the duration $\tau$ of atom-field interaction are adjusted so that $\Delta_1 − \Delta_2 \ll \pi/\tau$, the case of two-photon
resonance $\Delta_1 = \Delta_2 = \Delta$ can be realized with the following effective Hamiltonian

$$H_{eff} = \sum_{i=1,2} \frac{\hbar g_i^2}{\Delta_i} a_i^\dagger a_i \sigma_{ii} + \frac{\hbar g_1 g_2}{\Delta} \left( a_1^\dagger a_2 \sigma_{12} + a_1 a_2^\dagger \sigma_{21} \right).$$

(2.31)

Here $a_i$ and $a_i^\dagger$ are the annihilation and creation operators of the $i$th mode, while $\sigma_{ij} = |i\rangle \langle j|$ is the corresponding transition operator of $\Lambda$-atom. The couplings of the atom to the two modes are determined by the spatial mode functions $g_1(x) = g_{01} \sin(k_1 x)$ and $g_2(y) = g_{02} \sin(k_2 y)$, where $g_{0i} = E_0 \tilde{e}_i \langle i | \tilde{d} | 3 \rangle$, $i = 1, 2$ and $k_i$ is the wave vector of $i$th mode. $E_0$ is the so-called electric field per photon. $\tilde{e}_1$ and $\tilde{e}_2$ are the polarization vectors, while $\langle 1 | \tilde{d} | 3 \rangle$ and $\langle 2 | \tilde{d} | 3 \rangle$ are the dipole moments of the corresponding atomic transitions. The last term in the expression (2.31) describes the connection between the two interaction channels due to two-photon (Raman) transitions between $|1\rangle$ and $|2\rangle$ levels. When the condition of the Raman resonance is not carried out and contribution of the Raman transitions may be neglected, the interaction Hamiltonian reads as

$$H_{eff} = \sum_{i=1,2} \frac{\hbar g_i^2}{\Delta_i} a_i^\dagger a_i \sigma_{ii}.$$ 

(2.32)

The initial position distribution of atoms at the entrance of cavities is assumed to be Gaussian, i.e.

$$|f(x, y)|^2 = \frac{1}{2\pi \Delta x \Delta y} e^{-\frac{(x - \langle x \rangle)^2}{2(\Delta x)^2}} e^{-\frac{(y - \langle y \rangle)^2}{2(\Delta y)^2}}$$

(2.33)

with the widths $\Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$ and $\Delta y = \sqrt{\langle (y - \langle y \rangle)^2 \rangle}$ centered at the nodes of both waves, while the initial atomic states is considered in general as $a |1\rangle + b |2\rangle$ with $a$ and $b$ being the weights of the atomic lower states in the coherent superposition. The cavity modes are assumed to be initially in two-mode coherent state

$$|\text{field}\rangle = |\alpha_1\rangle_1 |\alpha_2\rangle_2, \ |\alpha\rangle_i = e^{-N/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle_i,$$

(2.34)
where \( |n\rangle_i \) are the Fock states for \( i\)th mode and \( N = |\alpha|^2 \). In this case, the state vector of this system at time \( t \) will have the form

\[
|\Psi(t)\rangle = \int dx dy \sum_{i=1,2} \sum_{n,m=0}^{\infty} \Phi^{(i)}_{n,m}(x, y, t) |n\rangle_1 |m\rangle_2 |i\rangle |x, y\rangle,
\]

(2.35)

where the amplitudes \( \Phi^{(i)}_{n,m}(x, y, t) \) can be derived from the following equation and its corresponding initial conditions

\[
\begin{aligned}
&i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H}_{eff} |\Psi(t)\rangle \\
&|\Psi(t = 0)\rangle = \int dx dy f(x, y) |x, y\rangle \otimes (a|1\rangle + b|2\rangle) \otimes |\alpha_1\rangle_1 |\alpha_2\rangle_2
\end{aligned}
\]

(2.36)

by substituting \( |\Psi(t)\rangle \) in (2.36) with the expression (2.35). In the case of two-photon resonance the amplitudes are written as

\[
\begin{aligned}
\Phi^{(1)}_{n,m}(x, y, t) &= f(x, y) \times \left\{ a \cdot C_{n,m} + G_1 \cdot \left( \exp \left[ -\frac{it}{\Delta} \Omega_{n,m+1} \right] - 1 \right) \right\} \\
\Phi^{(2)}_{n,m}(x, y, t) &= f(x, y) \times \left\{ b \cdot C_{n,m} + G_2 \cdot \left( \exp \left[ -\frac{it}{\Delta} \Omega_{n+1,m} \right] - 1 \right) \right\}
\end{aligned}
\]

(2.37)

(2.38)

where

\[
\begin{aligned}
G_1 &= \frac{g_1 g_2 \sqrt{n(m+1)}}{\Omega_{n,m+1}} C_{n-1,m+1} \cdot b + \frac{g_1^2 n}{\Omega_{n,m+1}} C_{n,m} \cdot a, \\
G_2 &= \frac{g_1 g_2 \sqrt{(n+1)m}}{\Omega_{n+1,m}} C_{n+1,m-1} \cdot a + \frac{g_2^2 m}{\Omega_{n+1,m}} C_{n,m} \cdot b,
\end{aligned}
\]

(2.39)

(2.40)

\( \Omega_{n,m} = g_1^2 n + g_2^2 m \) is the position-dependent Rabi frequency and

\[
C_{n,m} = e^{-|\alpha_1|^2 + |\alpha_2|^2}/2 \frac{\alpha_1^n \alpha_2^m}{\sqrt{n!m!}}.
\]

(2.41)

The functions \( \Phi^{(i)}_{n,m}(x, y, t) \) describe amplitude distribution for the position
of atom in the \((x, y)\) plane. They are proportional to the atomic initial position distribution and also display not trivial spatial features due to position dependent phase shifts acquired by \(\Lambda\) - atom passing through the standing waves.

It is well known \([7, 8]\) that the measurement of the phase shift of the cavity fields can be interpreted as a quantum spatial localization of the atom. In this way, below we investigate deflection of atoms with simultaneous quadrature measurement of the field. We have studied this problem in details for one-dimensional atomic deflection as well as for two-dimensional case in the dispersive limit assuming negligibly small the contribution of the Raman transitions in the system. In this case, there is no exchange of energy between the field and the atom, i.e. the interaction does not change the internal atomic state. This situation is cardinally changed for the \(\Lambda\) - atom under two-photon resonance condition. Indeed, in this case both amplitudes \(\Phi_{n,m}^{(1)}(x, y, t)\) and \(\Phi_{n,m}^{(2)}(x, y, t)\) corresponding to two atomic states \(|1\rangle\) and \(|2\rangle\) govern the spatial distribution of deflected \(\Lambda\) - atoms. Thus, the interaction between atom and intracavity field beside the initial position of the atom depends also on its internal states. The latter allows visualization of the atomic coherence in final position distribution of the atoms.

2.3.2 Probing and visualization of the atomic coherence by the deflection patterns

Position distributions

In this subsection, we focus on studies of position distributions assuming that the cavity modes are in given reference state. Two schemes of joint measurements have been proposed up to now, based on a quadrature measurement of standing wave fields as well as on a measurement of the phase states \([7, 8]\). We start by considering the case of the quadrature measurement, the other case will be considered in the section 2.3.3. To realize this we consider
implementation of a quadrature measurement on the field and use the following expression for the field quadrature state

$$|\chi_\theta\rangle = (2\pi)^{-1/4} e^{-(a^\dagger e^{i\theta} - \chi_\theta)^2/2 + \chi_\theta^2/4} |\text{vac}\rangle.$$  

(2.42)

Here, parameter $\theta$ is an angle characterizing the one-mode field quadrature in Wigner plane, $\chi_\theta$ is the corresponding eigenvalue and $|\text{vac}\rangle$ denotes the vacuum state with zero photon in the cavity. Here we are interested in calculation of the probability of finding an atom at position $(x, y)$ provided that a measurement of the two field modes with angles $\theta_1$ and $\theta_2$ is performed.
Using the reference state as $|\chi_{\theta_1}\rangle_1 |\chi_{\theta_2}\rangle_2$ where quadrature state $|\chi_{\theta_i}\rangle_i$ corresponds to the operator $a_i$, we obtain the joint probability as

$$W(\chi_{\theta_1}, \chi_{\theta_2}, x, y) = \sum_{i=1,2} |1 \langle \chi_{\theta_1} | 2 \langle \chi_{\theta_2} | \langle i | \langle x, y | \Psi(t) \rangle | 1 \rangle_1 | 2 \rangle_2|^2$$

$$= \sum_{i=1,2} \left| \sum_{n,m=0}^{\infty} \Phi^{(i)}_{n,m}(x, y, t) \langle \chi_{\theta_1} | n \rangle_1 \langle \chi_{\theta_2} | m \rangle_2 \right|^2. \quad (2.43)$$

We obtain the general expression for the factor $\langle \chi_{\theta} | n \rangle$ using the formula (2.42). This matrix element can be calculated as follows

$$\langle \chi_{\theta} | n \rangle = \frac{1}{\pi} \int \langle \chi_{\theta} | \alpha \rangle \langle \alpha | n \rangle d^2 \alpha, \quad (2.44)$$

where

$$\langle \chi_{\theta} | \alpha \rangle = e^{-|\alpha|^2/2} \cdot e^{-(\alpha e^{-i\theta} - \chi_{\theta})^2/2 + \chi_{\theta}/4} \quad (2.45)$$

and

$$\langle \alpha | n \rangle = e^{-|\alpha|^2/2} \cdot \frac{(\alpha^*)^n}{\sqrt{n!}}. \quad (2.46)$$

Similarly, the factors corresponding to the states $|\chi_{\theta_1}\rangle_1$ and $|\chi_{\theta_2}\rangle_2$ can be calculated.

In the following, we show how various initial atomic states in the form $a|1\rangle + b|2\rangle$, $|a|^2 + |b|^2 = 1$, with definite quadrature measurement can change the joint probability. Some typical results are depicted in Fig.2.12 as 2D distributions for the various superposition states $a|1\rangle + b|2\rangle$ as well as for the joint probabilities in 3D representation. The examples are taken for the measurement angles $\theta_1 = \theta_2 = 0$, $\alpha_1 = \alpha_2 = 2$, $\chi_{\theta_1} = \chi_{\theta_2} = 4$ and $\Delta x = \Delta y = 0.2\lambda_1$ ($\lambda_1 = 2\pi/k_1$). Considering 2D position distributions at the fixed parameters $\theta_1 = \theta_2$, $\alpha_1 = \alpha_2$, we conclude that the atomic distributions are turned in $x - y$ plane around the center $x = y = 0$ in dependence on the coefficients $a$ and $b$. The figures showing these features are presented in the Fig.2.12(a-f).
As our analysis shows, the orientation of the distribution changes with the value of the fraction $a/b$ when it is varying from $-\infty$ to $\infty$ and any given orientation corresponds a particular value of $a/b$. In general, the fraction $a/b$ is a complex number and we can represent it as $(a/b)_{\text{real}} \cdot e^{i\varphi}$, where $\varphi \in (-\pi/2, \pi/2)$ and $(a/b)_{\text{real}} \in (-\infty, \infty)$. It turns out that the additional factor $e^{i\varphi}$ does not change the orientation of the spatial distribution but changes the level of its stretching out in the direction of orientation. Thus, the distributions corresponding to $(a/b)_{\text{real}}$ and $(a/b)_{\text{real}} \cdot e^{i\varphi}$ are qualitatively the same. This fact provides a possibility to calculate the $(a/b)_{\text{real}}$ fraction from a given final atomic position distribution.

Thus, we demonstrate on Fig.2.12 that atomic superposition state can be qualitatively probed in two-dimensional patterns of deflected atoms. Indeed, the distribution corresponding to the superposition state $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ (Fig.2.12(e)) is turned on $\pi/4$ relatively to the position distribution of $\Lambda$ atoms that are initially in the state $|1\rangle$ (Fig.2.12(a,f)).

As our analysis shows, for higher values of the parameters $\alpha_1$ and $\alpha_2$, than those that have been used in our example, the form of distributions are essentially the same. The slight differences are related to additional relatively small structures that do not change the spatial orientation features of the distribution. Thus, obtained results are applicable also for more intensive electromagnetic fields. The choice of particular values $\theta_1 = \theta_2 = 0$ of the parameters $\theta_1$ and $\theta_2$ are conditioned by the explicitness of effects described above. The above results are mainly valid also for the other values of measurement angles $\theta_1$ and $\theta_2$ although differ in details from the results presented in Figs.2.12.

**Position distribution in the far-diffraction zone**

For the completeness of our analysis, we added in this section calculations for the position distribution of atoms in the far-field diffraction zone. This analysis is important for an experimental verification of the obtained effects.
It allows us to consider the obtained effects for realistic experimental conditions and hence defining the spatial resolution necessary for detection of these effects.

To modify previous calculations we use the expression for the free Hamiltonian of atoms \( H_{\text{free}} = \frac{\hat{p}^2}{2m} \) and rewrite the state vector of system at time \( t + t_0 \), where \( t \) is the free propagation time, in the following form

\[
|\Psi(t + t_0)\rangle = \exp\left[-\frac{i}{\hbar} H_{\text{free}} t\right] |\Psi(t_0)\rangle = \int dx dy \sum_{n,m=0}^{\infty} \Phi_{n,m}^{(i)}(x, y, t) |n\rangle_1 |m\rangle_2 |i\rangle |x, y\rangle,
\]

where

\[
\Phi_{n,m}^{(i)}(x, y, t) = -\frac{i M}{2\pi \hbar t} \int dx' dy' \Phi_{n,m}^{(i)}(x', y', t_0) \times \exp\left[\frac{i M}{2\hbar t} \left((x - x')^2 + (y - y')^2\right)\right].
\]

From (2.47) it is clear that for calculations of the position distribution at far-field diffraction zone we can use the expression (2.43) just by replacing \( \Phi_{n,m}^{(i)}(x, y, t) \) to \( \Phi_{n,m}^{(i)}(x, y, t) \). The results of numerical calculations of the distributions at near- and far-field regions are depicted in Figs. 2.13 showing a slight difference between them. The distribution in Fig. 2.13(b) corresponds to distance \( L = vt = 50 cm \) from the exit of cavity, where \( v \) is the velocity of atomic mass center in \( z \) direction and for its value we have chosen 100m/s.

**Momentum distributions**

The other approach analysing the atomic deflection process concerns to momentum distributions of atoms. In this section, we shortly discuss the momentum distribution in the deflected patterns of the Gaussian atomic wavepacket assuming the width of the wavepacket to be much smaller than the
wavelength of the modes. We calculate the probability of finding an atom with the transverse momentum \((p_x, p_y)\) provided that a measurement of the two field modes with angles \(\theta_1\) and \(\theta_2\) is performed. This procedure is similar to the calculation of conditional position distribution made above. Thus, momentum distribution is written as

\[
P(\chi_{\theta_1}, \chi_{\theta_2}, p_x, p_y) = \sum_{i=1,2} |1 \langle \chi_{\theta_1} | 2 \langle \chi_{\theta_2} | \langle i | p_x, p_y | \Psi(t) \rangle |^2
\]

\[
= \sum_{i=1,2} \sum_{n,m=0}^{\infty} \tilde{\Phi}_{n,m}^{(i)}(p_x, p_y, t) \langle 1 \chi_{\theta_1} | n \rangle_1 \langle 2 \chi_{\theta_2} | m \rangle_2^2
\]

where the amplitude in the momentum space is calculated by the Fourier transformation over spatial variables

\[
\tilde{\Phi}_{n,m}^{(i)}(p_x, p_y, t) = \frac{1}{2\pi} \int \int dx dy \Phi_{n,m}^{(i)}(x, y, t) \exp[-\frac{i}{\hbar}(p_xx + p_yy)].
\]

We illustrate the momentum distributions for narrow initial atomic wave
Figure 2.14: The momentum distribution $P(p_x, p_y)$ for $\Delta x = \Delta y = 0.2\lambda_1$, $\lambda_1 = 2\pi/k_1$. Brighter colors indicate higher values of distribution. The initial internal states of the atom are: (a) $a = -1/\sqrt{2}$ and $b = 1/\sqrt{2}$, (b) $a = 1/\sqrt{2}$ and $b = 1/\sqrt{2}$. We use the dimensionless momenta scaled in units of $\hbar k$. packets which obviously corresponds to a wider distribution in the momentum space. As shows analysis, in contrast to the position distributions the dependence of momentum distributions on initial internal states is not so evident, but in some cases it is still possible to find visible relations between distribution and initial internal state. The momentum distributions for two cases of initial atomic superposition states with the coefficients $a = -1/\sqrt{2}$, $b = 1/\sqrt{2}$ and $a = 1/\sqrt{2}$, $b = 1/\sqrt{2}$ are shown in Fig.2.14. Comparing Fig.2.14(a) and Fig.2.14(b) we realize that ranges of localizations of the momentum for these cases are stretched in perpendicular directions, i.e. the distribution corresponding to the superposition state $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ is turned on $\pi/2$ relatively to the momentum distribution of $\Lambda$ - atoms that are initially in the state $\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$. Comparing the results of momentum distributions with corresponding position distributions in Fig.2.12(b) and Fig.2.12(e) we conclude that they are in accordance with uncertainty relations.
2.3.3 Evidence of Raman resonance in atomic deflection

In this section we illustrate the principal differences between the cases of interaction in Raman resonance and ordinary off-resonance from the point of view of an atomic beam deflection by two crossed standing wave. In this way, we derive the amplitudes $\Phi_{n,m}^{(i)}(x, y, t)$ in (2.35) for interaction Hamiltonian (2.32). For atomic initial internal state $a|1\rangle + b|2\rangle$ the amplitudes are written as

$$\Phi_{n,m}^{(1)}(x, y, t) = f(x, y)C_{n,m} \exp \left[ -i \frac{g_1^2 t}{\Delta_1 n} \right] \cdot a,$$

$$\Phi_{n,m}^{(2)}(x, y, t) = f(x, y)C_{n,m} \exp \left[ -i \frac{g_2^2 t}{\Delta_2 m} \right] \cdot b.$$ 

(2.51)

Considering different regimes of interaction of $\Lambda$-atom by using the spatial amplitude (2.37), (2.39) and (2.51) we demonstrate that the deflection patterns are essentially different for the case of one-photon and two-photon interactions. The most convenient approach to realize our goal is the investigation of conditional position distribution of atoms provided that the two mode field is in a given phase state

$$|\Psi_R\rangle = |\varphi_1\rangle_1 |\varphi_2\rangle_2,$$ 

(2.52)

where

$$|\varphi\rangle_i = \frac{1}{\sqrt{2\pi}} \sum_n e^{-i\varphi_n} |n\rangle_i.$$ 

(2.53)

The discussing distribution is

$$W(x, y) = \sum_{i=1,2} |\langle x, y | \langle i | \langle \Psi_R | \Psi(t) \rangle |^2.$$ 

(2.54)
Using the formulas (2.35) and (2.51-2.54) we obtain the following expression for $W(x, y)$

$$W(x, y) = \frac{1}{(2\pi)^2} \left| \sum_{i=1,2} \sum_{n,m} \Phi_{n,m}^{(i)}(x, y, t) \cdot e^{-i(n\varphi_1 + m\varphi_2)} \right|^2. \quad (2.55)$$

Obviously, distributions (2.51) and (2.55) crucially depend on phases $\varphi_1$ and $\varphi_2$ which concretize the ranges of atomwaves interactions. We choose $\varphi = 0$, which gives the best spatial localization of the scattered atoms. Thus, the probability $W(x, y, \varphi = 0) = W(x, y)$ describes the selected scattering events of only those atoms that have passed the nodes of the field with the spatial mode function $\sin(k_1 x), \sin(k_2 y)$. As it was shown earlier, such a joint measurement procedure plays a role of a spatial filter. Experimentally, this procedure can be realized by a mechanical slit or mask placed in front of a node of the electromagnetic field. For detailed discussion of this point, see [7].

The results of concrete calculations based on Eq. (2.55) are depicted on the Fig. 2.15 and Fig. 2.16 for the different regimes of interaction. In Fig. 2.16 the position distribution functions $W(x, y)$ are depicted in the case of two-photon resonant interaction and for two different choices of the initial coherence of the internal atomic state. As we can see, these results are similar to those in the Fig. 2.12(a1,e1) obtained for the quadrature-measurement scheme. The case of different detunings leading to one-photon atomic transitions is depicted in Fig. 2.15. Comparing two types of the graphics allows us to make clear the peculiarities of coherent Raman processes in comparison with corresponding one-photon processes from the point of view of atomic optics.

Let us consider probability distributions depicted in Fig. 2.15(a) and Fig. 2.16(a) that correspond to the situation when atoms enter cavity being initially in the lower state $|1\rangle$. As we see, the position distribution in Fig. 2.15(a) shows high spatial localization of deflected atoms in $x$ direction, while in $y$
Figure 2.15: The position distribution function $W(x, y)$ for the case of $\Delta_1 \neq \Delta_2$, $\Delta x = \Delta y = 0.3\lambda_1$, $\lambda_1 = 2\pi/k_1$. Initial state of the atom is described by (a) $a = 1$ and $b = 0$; (b) $a = 1/\sqrt{2}$ and $b = 1/\sqrt{2}$.

direction their behaviour have remained unchanged, i.e. determined by the initial Gaussian distribution. In other words, the impact of the initial field state is the one-dimensional distribution of the atoms. It could be anticipated, because in this case the atom and the field interact through only one channel, (it is the interaction between first mode of the field and $|1\rangle \leftrightarrow |3\rangle$ transition), as the atom due to off-resonant nature of the interaction always remains in the state $|1\rangle$. In contrast to that, in the case of two-photon resonance the two-photon $|1\rangle \leftrightarrow |2\rangle$ transition leads to activation of the interaction between the second mode of the field and the $|2\rangle \leftrightarrow |3\rangle$ transition. As a result, the structure of the position distribution in Fig.2.16(a) has acquired some features indicating this fact: The distribution showing the localization near some points in both $x$ and $y$ directions is described by two-dimensional patterns. Particularly, it is expressed by emergence of four additional walls forming a structure resembling two closed cycles (see, Fig.2.16(a)).

If the atom enters the cavity in a superposition of two lower $|1\rangle$ and $|2\rangle$ states, the impact of the field on atomic position distributions contains two-dimensional localization patterns even in case of the absence of Raman transitions (see Fig.2.15(b) and Fig.2.16(b)). The difference between the distributions of considered two regimes becomes obvious when we analyze the
Figure 2.16: The position distribution function $W(x, y)$ for the case of $\Delta_1 = \Delta_2$, $\Delta x = \Delta y = 0.3\lambda_1$, $\lambda_1 = 2\pi/k_1$. Initial state of the atom is described by (a) $a = 1$ and $b = 0$; (b) $a = 1/\sqrt{2}$ and $b = 1/\sqrt{2}$.

graphics in the following manner. If Raman resonance is violated the position distribution (see, Fig.2.15(b)) has described by two perpendicular planes of symmetry passing through $x$ and $y$ axes. It expresses the fact that the impact of interaction between the first mode and $|1\rangle \leftrightarrow |3\rangle$ transition and the impact of interaction between the second mode and $|2\rangle \leftrightarrow |3\rangle$ transition are independent. They depend only on populations on the corresponding atomic levels, which in this regime, at the beginning of interaction are equal to $|a|^2$ and $|b|^2$ respectively, (see Eq.(2.31)). In the two-photon resonance regime $|1\rangle \leftrightarrow |2\rangle$ transition destroys this independence and hence, leads to breaking of the symmetry. This fact is evidently reflected in Fig.2.16(b) where $x$ and $y$ are not axes of symmetry for the distribution.

2.4 Formation of two-dimensional nano-structures in atomic deflection

In this section, we discuss formation of two-dimensional spatial structures of atoms within the optical wavelength due to atomic diffraction on two crossed standing electromagnetic fields in optical cavities. This analysis proposed
for $V$-type atomic configuration under dispersive atom-field interactions in quantum regime. Here, we demonstrate that localization procedure for narrow initial position distributions of atoms leads to production of controllable two-dimensional pattern structures with feature spacing smaller than a wavelength of light in a cavity. In this way, we calculate the atomic deflection patterns in the transverse plane to the direction of the center of atomic mass motion that is accompanied by the quadrature measurement of the fields. This procedure means that the position distribution of atoms is studied assuming that the cavity modes are in a given reference state. Two limited cases of the reference field states: phase- and amplitude-states of standing waves are considered. We analyze the probability to determine two-dimensional position of atoms with maximum probability for a given measurement that result as necessary condition for extremes. We show that multiple simultaneous quadrature measurements allow increasing localization precision in formation of atomic structures. The other part of this paper is devoted to the problem of atomic spatial localization in the presence of entanglement of two light beams. Note, that spatially entangled atomic deflections resulting from interaction of a $V$-level atomic beam with a two-mode quantized cavity field have been considered [28, 157]. For concrete calculations, as an entangled two-mode state, we choose entangled pair-coherent state with an equal photon number in both the modes [148, 158]. Depending on the atom-light interaction scheme, various two-dimensional patterns in a sub-wavelength regime for $V$-type atomic configurations are reported. We believe that this approach could be used for the formation of atomic nano-structures and quantum lithography.
2.4.1 Atom-Field Interactions and quadrature measurements

The scheme proposed for production of atomic nanostructures consists of two crossed one-mode standing waves and atomic beam crossing the interaction region (see, Fig. 2.17(a)). For concrete calculations we consider a three-level atom with a $V$-type configuration of energy levels (see, Fig. 2.17(b)) moving along the $z$ direction and passing through a cavity electromagnetic field. We investigate the position patterns of deflected atoms in the $x - y$ plane assuming that the initial transverse distribution of atoms is Gaussian. We concentrate on the interaction of a three-level atom, initially prepared in a ground atomic state with two quantized cavity modes in coherent states $|\text{field}\rangle = |\alpha_1\rangle_1 |\alpha_2\rangle_2$. The interaction Hamiltonian describing such a system reads as follows:

$$\hat{H} = \sum_{i=1,2} \hbar \frac{|g_i|^2}{\Delta_i} \hat{\sigma}_{33} \hat{a}_i^\dagger \hat{a}_i,$$

(2.56)

where $g_1(x) = g_{01} \cos(k_1 x + \xi_1)$, $g_2(y) = g_{02} \cos(k_2 y + \xi_2)$ are spatially-dependent atom-standing wave coupling constants, $\Delta_1$ and $\Delta_2$ are the detunings, $\hat{a}_i^\dagger$ and $\hat{a}_i$ are the annihilation and creation operators of the $i$-th mode and $\hat{\sigma}_{ij} = |i\rangle \langle j|$. We choose the initial state of the combined system as

$$|\Psi(0)\rangle = \int dx dy f(x, y) |3\rangle |\alpha_1\rangle_1 |\alpha_2\rangle_2 |x\rangle |y\rangle.$$

(2.57)

In this case, the state of the combined system after the interaction will be calculated in the following form:

$$|\Psi(t)\rangle = \int dx dy f(x, y) |3\rangle |\alpha_1 e^{i\phi_1(x)}\rangle_1 |\alpha_2 e^{i\phi_2(y)}\rangle_2 |x\rangle |y\rangle.$$

(2.58)

where

$$\phi_1(x) = -\frac{g_{01}^2}{\Delta_1} \cos^2(k_1 x + \xi_1), \quad \phi_2(x) = -\frac{g_{02}^2}{\Delta_2} \cos^2(k_2 y + \xi_2)$$

(2.59)
and \( f(x, y) \) is the initial wave function of atom. Below, for concrete calculations, we consider atomic beam with spatial Gaussian distribution

\[
|f(x, y)|^2 = |f(x)|^2 |f(y)|^2 = \frac{1}{2\pi\Delta x\Delta y} e^{-\frac{(x-\langle x \rangle)^2 + (y-\langle y \rangle)^2}{2(\Delta x)^2}}
\]

(2.60)

where \( \Delta z = \sqrt{\langle (z - \langle z \rangle)^2 \rangle} \) and \( z = x, y \).

It is obvious that due to non-resonant interaction the atom remains in the initial ground state during time-evolution as is indicated in (2.58). The dispersive interaction only changes the phases of the coherent states of standing waves by the functions \( \phi_1(x) \) and \( \phi_2(y) \), which depend on the positions \( x \) and \( y \) of the center-of-mass of the atom in the corresponding cavity. The result (2.58)-(2.59) relied to two-dimensional case [27] is a generalization of well-known one for one-dimensional case [8].

![Figure 2.17: (a) The setup proposed for production of atomic nanostructures consisting of two cavities and an atomic beam crossing the interaction region. (b) Energetic levels of a \( V \)-type atom interacting with two cavity modes with coupling constant \( g_1 \) and \( g_2 \).](image)

The two-dimensional localization scheme relies on measuring the states of the intracavity fields. For this, following the cited papers we make use phase-sensitive quantum state such as quadrature field eigenstate

\[
|\chi_\theta\rangle = (2\pi)^{-1/4} e^{-(\hat{a}^\dagger e^{i\theta} - \chi_\theta)^2/2 + \chi_\theta^2/4} |\text{vac}\rangle,
\]

(2.61)
where $\theta$ parameter is an angle in the Wigner plane characterizing the quadrature and $\chi_\theta$ is the corresponding eigenvalue of this state. According to the procedure of reconstruction of atomic position, we investigate the probability for atom to be at position $x$ and $y$ assuming that two field measurements have the results $\chi^{(1)}_\theta$ and $\chi^{(2)}_\theta$. The probability is given as

$$P(x, y, \chi^{(1)}_\theta, \chi^{(2)}_\theta) = \left| \langle x, y | \langle \chi^{(1)}_\theta, \chi^{(2)}_\theta | \Psi(t) \rangle \right|^2. \quad (2.62)$$

In the dispersive limit of atom-field interaction the joint position distribution to find the atom at position $(x, y)$ provided that the quadrature amplitudes of the electromagnetic field modes equal to $\chi^{(1)}_\theta$ and $\chi^{(2)}_\theta$ is represented as a product of two independent probability distributions for $x$ and $y$ respectively

$$P(x, y, \chi^{(1)}_\theta, \chi^{(2)}_\theta) = |f(x, y)|^2 P^{(1)}_\alpha(\chi^{(1)}_\theta, x) P^{(2)}_\alpha(\chi^{(2)}_\theta, y). \quad (2.63)$$

Here, the first factor is the initial position distribution, while the other two are conditioned by interaction between the atom and two intracavity field modes. Each of them can be presented as

$$P^{(1)}_\alpha(\chi_\theta, x) = \left| \langle \chi_\theta | \alpha e^{i \phi t} \rangle \right|^2 = \frac{1}{\sqrt{2\pi}} e^{-2(\alpha_R - \chi_\theta/2)^2}, \quad (2.64)$$

where

$$\alpha_R = \text{Re}(\alpha e^{i(\phi t - \theta)}). \quad (2.65)$$

The position distribution (2.64) depends on the intensity of field via $\alpha$, the coupling constant of atom-field interaction, its duration and detuning via $g_0^2 t / \Delta$ and measured values for $\chi_\theta$. Thus, measurement procedure acts as spatial filters and can considerably narrow the passing atomic beam. For completeness, we also show on Fig. 2.18 the probability $P_\alpha(\chi_\theta, x)$ as a function of $x$ and the value of quadrature amplitude $\chi_\theta$ in accordance with the well known result [8].
2.4.2 Spatial Localization And Two-Dimensional Structures

At first, we briefly discuss the results on atomic localization in one-dimensional case. Thus, we consider the interaction between the atom and one of the standing wave, which results in deflection on $x$ direction. In this way, we follow the cited papers [8, 27]. The novelty of our consideration is that some details of measurement-induced atomic localization are made clear and simple applications to two-dimensional case are presented.

It is easy to find the ranges of position for maximum probability $P_\alpha(\chi_\theta, x)$ for a given measurement result $\chi_\theta$. For given result of the measurement the atom most probably located at positions $x$, which can be obtained as a necessary condition for extreme of (2.63) as a solution of the equation

$$\chi_\theta = 2\alpha R(x),$$

which in the detailed form reads as

$$\cos \left( \gamma \cos^2(kx + \xi) \right) = \frac{\chi_\theta}{2\alpha},$$

where $\gamma = -g_0^2t/\Delta$. To explain the procedure of the effective filtering we
present on Figs. 2.19 the curve (2.66), showing dependence $\chi_\theta$ on the position $x$, at the same time with the position distributions. Thus, for illustration, we also add typical distributions of initial and diffracted atomic beams, which are presented correspondingly by dashed and dot-dashed curves. In this way, two limiting cases of spatial resolution are depicted in Fig. 2.19(a) and Fig. 2.19(b). The case of strong atomic localization is realized on Fig. 2.19(a) if initial atomic distribution is located between node and antinode of standing wave and corresponds to the measured value $\chi_\theta = 2$. The opposite regime of weak localization is realized if initial atomic distribution is located at antinode of the standing wave. Our analysis shows that the maxima of final distribution are situated at intersection of the line of value $\chi_\theta$ with the curve $2\alpha_R(x)$. For the completeness, we also calculate the widths of strong and weak atomic localizations.

The position distribution of diffracted atoms in the ranges of its maxima can be approximated by Gaussian as

$$P_\alpha(\chi_\theta, x) \approx \frac{1}{\sqrt{2\pi}a^2} e^{\frac{(x-x_0)^2}{a^2}}, \quad (2.68)$$

\[\text{Figure 2.19: Comparative analysis of the extreme curve (in dependence of the dimensionless position, solid wavelike curve) with both initial atomic distribution (dashed curve) and the distribution (dot-dashed curves) for the parameters as on the Fig. 2.18}\]
where the width $\Delta x$ is defined by

$$\frac{1}{a} \approx \sqrt{2k\alpha} \left| \frac{g_0^2 t}{\Delta} \sin(2kx_0 + 2\xi) \sin(\phi(x_0)t - \theta) \right|.$$  (2.69)

Finally, using approximate values for the trigonometric functions it can be estimated as

$$a = \frac{1}{\sqrt{2\alpha k}} \left| \frac{g_0^2 t}{\Delta} \right|^{-1}.$$  (2.70)

It should be mentioned that the approximations made off in the last three expressions are reasonable only if $\sin(2kx_0 + 2\xi) \sim 1$ and $\sin(\phi(x_0)t - \theta) \sim 1$ conditions are satisfied. This conditions otherwise can be considered as the restrictions for values of $\chi_\theta$ to be not too close to the parameter $\pm 2\alpha$. This limiting case shown in Fig. 2.19(b) is approximately calculated as follows

$$P_\alpha(\chi_\theta, x) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-x_0)^2}{b^2}},$$  (2.71)

where

$$b \approx \frac{1}{k} \left( \frac{2}{\alpha^2} \right)^{\frac{1}{8}} \left| \frac{g_0^2 t}{\Delta} \right|^{-\frac{1}{2}}.$$  (2.72)

Note, that for a typical parameters of atom-field interactions, $b > a$.

Thus, we conclude that by varying the position of initial beam and measuring field quadrature states both narrow- and wide- localized structures can be formed. The above described feature allows us to shape the atomic beam in different ways and not only in one- but also in two-dimensions. The last case it is possible to realize by performing localization procedure for both dimensions. Some typical examples showing the two-dimensional spatial localization of initial atomic distribution are depicted in Figs. 2.20. The initial Gaussian distribution is shown on Fig. 2.20(a). The Fig. 2.20(b) describes the situation when the initial atoms are located between node and antinode of crossed standing waves for both directions and quadrature measurements.
are performed with the results: $\chi_1 = \chi_2 = 0$. The condition for atoms situated between node and antinode of modes are particularly realized, if $\chi_1 = \chi_2 = \pi/4$. In this case, the diffracted atomic beam is squeezed in both directions and forms a narrow peak. The distribution in Fig. 2.20(c) is formed by placing the entering beam at nodes of the standing waves in both $x$ and $y$ directions. To realize this case we need also to choose the appropriate values for measured quadratures. For this goal we turn to the above results on one-dimensional localization where we have distinguished two cases for measured values of quadrature, the first leading to structures with width $a$ and the second with $b$, bigger than $a$. According to this procedure, in the distribution (Fig. 2.20(c)) the measurement in $x$ direction squeezes the beam to have the wider width $b$, while in $y$ direction-the narrower width $a$. In the Fig. 2.20(d) another structure is shown, its formation requires locating the entering beam in $x$ direction at the node and in $y$ direction between the node and antinode of the standing wave. The measuring values of the quadrature in this case are $\chi_1 = 4$ and $\chi_2 = 0$.

### 2.4.3 Entanglement And Localization

It has been mentioned above that in the dispersive limit of atom-field interaction the time evolutions of spatial amplitudes on $x$ and $y$ in the state (2.58) are independent. Therefore, in this case two spatial channels contribute to the atomic scattering amplitude separately and hence the probability (2.63) is represented as a product of two independent probability distributions for $x$ and $y$, respectively. This situation is cardinally changed for the case of correlated modes [28]. In this section, we expand the results presented earlier [28] considering the procedure of simultaneous two quadrature measurements. For concrete calculations we use so-called pair-coherent states [148, 158] with an equal photon number in both modes $\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) |\zeta\rangle_p = 0$ that is described as eigenstates of the product of the annihilation operators of the
two radiation modes $a_1a_2|\zeta\rangle_p = \zeta|\zeta\rangle_p$. This state is represented as

$$|\zeta\rangle_p = N_0 e^{-|\zeta|} \int \frac{d\theta}{2\pi} \left| \sqrt{\zeta} e^{i\theta} \right>_1 \left| \sqrt{\zeta} e^{i\theta} \right>_2$$

(2.73)

where $N_0$ is a normalization constant. At first stage, we consider the joint quadrature measurements of pair-coherent states. The corresponding probability reads as

$$W(\chi^{(1)}_\theta, \chi^{(2)}_\theta) = \left| \langle \chi^{(1)}_\theta, \chi^{(2)}_\theta | \zeta \rangle_p \right|^2,$$

(2.74)

where

$$\langle \chi^{(1)}_\theta, \chi^{(2)}_\theta | \zeta \rangle_p = N_0 e^{-|\zeta|} \int \frac{d\eta}{2\pi} \langle \chi^{(1)}_\theta | \sqrt{\zeta} e^{i\eta} \rangle_1 \langle \chi^{(2)}_\theta | \sqrt{\zeta} e^{-i\eta} \rangle_2$$

(2.75)
and is depicted on Fig. 2.21. The result shows that two field measurements have the results $\chi^{(1)}_\theta$ and $\chi^{(2)}_\theta$ that are correlated on the definite range. The introduction of atoms in the system makes it too complicated for this type of analysis, nevertheless it is easy to anticipate that the correlation between modes will allow the formation of nonlinear structures. Indeed, as it is shown later in Fig. 2.22, due to the unique features of the intracavity field the outcome of the atomic scattering is completely different from the result presented in Figs. 2.20(b,c,d). Considering the atomic diffraction, we choose

The initial state of the combined system as

$$|\Psi(0)\rangle = N_0 e^{-|\zeta|} \int dx dy \int \frac{d\eta}{2\pi} f(x, y) \left| \sqrt{\zeta} e^{i\eta} \right>_1 \left| \sqrt{\zeta} e^{-i\eta} \right>_2 |3 \rangle |x\rangle |y\rangle. \quad (2.76)$$

Calculation of time-evolution leads to the following result for the probability to find the atom at positions $(x, y)$ if the quadrature measurements of two modes give the values $\chi^{(1)}_\theta$ and $\chi^{(2)}_\theta$

$$|\Psi(t)\rangle = N_0 e^{-|\zeta|} \int dx dy \int \frac{d\eta}{2\pi} f(x, y) \left| \sqrt{\zeta} e^{i(\phi_1(x)t+\eta)} \right>_1 \left| \sqrt{\zeta} e^{i(\phi_2(y)t-\eta)} \right>_2 |3 \rangle |x\rangle |y\rangle. \quad (2.77)$$
Now, using the expression (2.61) and (2.62) the atomic distribution for given measured quadratures is derived as

\[ P(x, y, \chi^{(1)}_\theta, \chi^{(2)}_\theta) = |f(x, y)|^2 P_\zeta(x, y, \chi^{(1)}_\theta, \chi^{(2)}_\theta), \]  

(2.78)

where

\[
P_\zeta(x, y, \chi^{(1)}_\theta, \chi^{(2)}_\theta) = N e^{-2|\zeta|^2} \left| \int dxdy \int \frac{d\eta}{2\pi} f(x, y) \left| \sqrt{\zeta} e^{i(\phi_1(x)t+\eta)} \right\rangle_1 \left\langle \chi^{(1)}_\theta \right| \sqrt{\zeta} e^{i(\phi_2(y)t-\eta)} \left\rangle_2 \right|^2 \]

(2.79)

and \( N \) is a new normalization factor.

It is not difficult to realize that the maxima of the deflection pattern adopt various types of curves in the \( x - y \) plane in dependence of the parameters of atom-fields interaction on one side and in dependence on location of the position of initial beam and measuring field quadrature states on the other side. Nevertheless, we restrict ourselves by presenting only two examples

![Figure 2.22: The conditional position distributions of atoms diffracted on standing wave in pair-coherent state for the parameters: (a) \( \theta = \pi/2, \xi_1 = \pi/2, \xi_2 = 0 \) and \( \chi_1 = \chi_2 = 4 \); (b) \( \theta = 0, \xi_1 = \xi_2 = \pi/2 \) and \( \chi_1 = \chi_2 = 4 \).](image)

\( k \) is the wave number of intracavity mode.

of atomic pattern with simple geometrical structures. In Fig.2.22 we show
two special results obtained for two different set of parameters: a circle for \( \theta = \pi/2 \) (phase), \( \xi_1 = \pi/2, \xi_2 = 0, \chi_1 = \chi_2 = 4 \) and a cross for \( \theta = 0 \) (amplitude), \( \xi_1 = \xi_2 = \pi/2, \chi_1 = \chi_2 = 4 \). These results expand the set of possible subwavelength elementary structures obtained by field quadrature measurements, providing opportunity for construction of a wider range of complex atomic structures.
Chapter 3

SUPERCONDUCTING QUBIT IN TIME-MODULATED FIELD

3.1 Introduction

Superconducting circuits based on Josephson junctions are promising candidates for studying fundamental physics and implementing qubits and controllable quantum two-level systems for quantum computing (see, for example, [50–53] for reviews). The simplest Josephson-junction ($JJ$) qubit consists of a small superconducting island with $n$ excess Cooper-pair charges connected by a tunnel junction with capacitance $C_J$ and Josephson coupling energy $E_J$ to a superconducting electrode and the single-electron charging energy $E_C$. In the case of a qubit only two charge states with $n = 0$ and $1$ play a role while all other charge states, having a much higher energy, can be ignored. Thus, a superconducting charge qubit [54] behaves as an artificial two-level atom in a Cooper box, which is well described by two charge states, and the electrostatic energy difference between these states is controlled by the normalized gate charge.
When a qubit is driven by an external periodically time-dependent electromagnetic field, it has given rise to new quantum effects such as Rabi oscillations and coherent control \([55-58]\), which are the bases for quantum operations. In a series of experiments many fundamental effects from quantum optics have been demonstrated \([59-65]\), including a lasing effect with a Josephson-junction charge qubit embedded in a superconducting resonator \([61]\). Superconducting qubits usually have short coherence time; therefore, to decrease the time for performing gate operations a large-amplitude external field should be applied. The dynamics of a qubit driven by large-amplitude external fields in the case of driving around the region of avoided level crossing has been also studied (see, \([66]\) and \([67]\) for reviews).

Most studies of qubit dynamics assume the driving field to be monochromatic or a single cavity mode. In the present work we investigate dynamics of a qubit and the phenomenon of Rabi oscillations for an artificial two-level atom interacting with a monochromatic field with time-modulated amplitude. Such an external field can be also presented as a bichromatic field that consists of two components of equal amplitudes which are symmetrically detuned from the qubit resonance frequency. In this case, the modulation frequency is displayed as the difference between frequencies of two spectral components. This approach, involving modulation of the energy splitting of a qubit in complicated form due to interaction with an external bichromatic field, is different from the standard scheme of laser physics in which the bichromatic field leads to dipole transitions between two states of atoms. This approach can be also applied for investigation of a wide variety of interesting phenomena including tunneling dynamics of time-dependently driven nonlinear quantum systems. In addition, this problem offers an ideal testing ground for studying the fundamental interactions between qubits.
and multi-spectral component light. Note, that the scheme of the Josephson-junction qubit considered in this chapter seems to be close to the experimental scheme on the frequency-modulated transmon qubit performed most recently in Ref. [68].

The other goal of this work is application of the method of quasienergies and quasienergetic states (QESs) (or the so-called Floquet states) for the qubit in a bichromatic field. Note that, at first, the QESs of the composite system consisting of an atom and time-periodic e.m. field have been considered in [69–72]. These states provide a classical counterpart to well-known atomic-dressed states [73] in which the coupling to the laser is described by a classical field, whereas the coupling to the vacuum must be described in second quantization. However, one may still hope that in the limit of a macroscopically relevant laser field, both approaches lead to the same results. On the other hand, a certain advantage of the classical treatment implied by the Floquet approach lies in the fact that laser pulses can be handled more easily than in a fully quantized approach to the field (see, e.g., [74]). In the Floquet picture the QESs of the composite system are formed in a strong external field, and the radiation processes and spectral lines are described by transitions between them due to the interaction of the composite system with an electromagnetic vacuum or with a weak probe field. In this way, the master equations in the QES basis were obtained in [75, 76] and in the dressed-state basis in [77]. Thus, the method of QESs is a powerful theoretical framework for the study of bound-bound multiphoton transitions driven by periodically time-dependent fields (see, for review [78]). There have been several experiments on nonlinear and quantum optics that have been interpreted in terms of quasienergy levels including basic experiments on the resonance fluorescence and the probe absorption spectroscopy for a two-level atom in a strong laser field. QESs and dressed states have also been used in areas of radiation corrections to atomic levels in the presence of a strong laser field, including
the calculation of the Lamb shift [159–165]. The dressed-state approach including atomic motion was introduced in Ref. [166], while the QES method was used for strongly confined ions in Refs. [167, 168] for multiphoton processes with laser-cooled and trapped ions, for the scheme of an ion-trap laser [169, 170], and for investigations of photons correlation in an ion-trap system [171].

Applications of QESs and quasienergies to Josephson qubits in a driving field have been done in several papers [67, 79–81], including a review paper on Landau-Zener-Stückelberg interferometry [67], probe spectroscopy of QESs [80], application of the Floquet theory to Cooper pair pumping [79], and observation of the Stark effect and generalized Bloch-Siegert shift in the experiment with a superconducting qubit probed by resonant absorption via a cavity [81]. The experiments on the Rabi oscillations in monochromatically driven Josephson qubits have been performed and interpreted on the basis of dressed states [56, 57].

QESs for a two-level atom in the bichromatic field have also been studied in a series of papers (see, for example, [76, 98–102]).

We believe that the results of forming atomic spectral lines with strongly different frequencies under bichromatic radiation are important also for the superconducting qubit inducing additional Rabi oscillations on quasienergetic states of the qubit. Additionally, we demonstrate below that quasienergetic states and quasienergies of the bichromatically driven superconducting qubit under consideration differ drastically from the analogous well-known states of the standard two-level atom in a bichromatic field, and due to this difference unusual field-dependence effects appear for the qubit.

In this chapter, we present analytical results for nontrivial dynamics of a qubit in a time-modulated field (a bichromatic field), particularly, considering in detail time-dependent populations of qubit states. We calculate QESs and quasienergies of the composite system "superconducting qubit plus time-modulated field" in resonance approximation by using the Furry
picture.

3.2 Furry picture for qubit in time-modulated field

The qubit is realized if the charging energy of a superconducting electron box is much larger than the Josephson coupling energy. In the regime of low-level excitation the system is formed by two charge states: $|\downarrow\rangle$ and $|\uparrow\rangle$ which have either zero Cooper pairs or one Cooper pair. Thus, the system that we consider here is a qubit coupled to a time-modulated field (or a bichromatic field) with the Hamiltonian

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_V, \quad (3.1)$$

where

$$\hat{H}_0 = -\frac{1}{2}(\varepsilon_0 + f(t))\hat{\sigma}_z, \quad \hat{H}_V = -\frac{\Delta}{2}\hat{\sigma}_x. \quad (3.2)$$

Here, the external field reads

$$f(t) = 2A \cos(\omega_0 t) \cos(\delta t), \quad (3.3)$$

where $\omega_0$ and $\delta$ are the central and modulation frequencies, provided that $\delta << \omega_0$. This external field can be presented as a bichromatic field of the form

$$f(t) = A[\cos(\omega_1 t) + \cos(\omega_2 t)] \quad (3.4)$$

with equal amplitudes of two spectral components at the frequencies $\omega_1 = \omega_0 - \delta$ and $\omega_2 = \omega_0 + \delta$.

Here, $\varepsilon_0 = E_Q(1 - 2n_g)$ is the electronic energy difference between the ground and excited states of the qubit and $\Delta = E_j$ is the Josephson coupling energy or the tunneling amplitude between the basis states. The operators $\hat{\sigma}_x$, $\hat{\sigma}_z$ denote the Pauli spin matrices: $\hat{\sigma}_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$, $\sigma_x = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|$. The Hamiltonian Eq. (3.1) describes various physical systems in addition to
the JJ artificial atom [116]. In general, it describes the tunneling dynamics of bichromatically driven nonlinear quantum two-level systems.

It should be noted that very often in area "atom+laser" interaction the other Hamiltonian is used, in which the coupling of a time-dependent electromagnetic field to the transition dipole moment between two states of atoms takes place in contrast to the case of a superconducting qubit, where an external field drives the atomic energetic levels Eqs. (3.1) and (3.2). The corresponding Hamiltonian \( \hat{H}_{at} \) describing interaction along the \( x \) axis can be related to the Hamiltonian Eq. (3.1) with the time-dependent component along the \( z \) axis by a rotation around the \( y \) axis. The result reads

\[
\hat{H}_{at} = e^{-i\frac{\pi}{4}\hat{\sigma}_y} \hat{H}(t) e^{i\frac{\pi}{4}\hat{\sigma}_y} = -\frac{1}{2} \Delta \hat{\sigma}_z - \frac{1}{2} (\varepsilon_0 + f(t)) \hat{\sigma}_x.
\] (3.5)

The later Hamiltonian is typical for a natural two-level atom interacting with a bichromatic field. In this case, the parameter \( \Delta \) describes an energy difference and the interaction term is responsible for the transitions between two atomic states.

We describe the dynamics of the system in the Furry-state representation \( |\Psi(t)\rangle = \hat{U}(t)|\Psi_U(t)\rangle \), in which the equation for the vector state of the full system is

\[
i \frac{\partial}{\partial t} |\Psi_U(t)\rangle = \hat{H}_I |\Psi_U(t)\rangle.
\] (3.6)

The interaction Hamiltonian is given by

\[
\hat{H}_I(t) = \hat{U}^{-1}(t) \hat{H}_V \hat{U}(t) = -\frac{\Delta}{2} \hat{U}^{-1}(t) \hat{\sigma}_x \hat{U}(t)
\] (3.7)

while the unitary operator \( \hat{U}(t) \) obeys the equation of motion

\[
i \frac{\partial}{\partial t} \hat{U}(t) = \hat{H}_0 \hat{U}(t).
\] (3.8)
Chapter 3. SUPERCONDUCTING QUBIT IN TIME-MODULATED FIELD

It is easy to realize that operator $\hat{U}(t)$ has a simple form

$$\hat{U}(t) = \exp \left[ -i \int_0^t \hat{H}_0(t') dt' \right] = \exp(i\varphi(t)\hat{\sigma}_z),$$

(3.9)

where

$$\varphi(t) = \frac{1}{2} \left[ \varepsilon_0 t + \frac{A}{\omega_1} \sin(\omega_1 t) + \frac{A}{\omega_2} \sin(\omega_2 t) \right].$$

(3.10)

Thus, the interaction Hamiltonian is calculated in the following form

$$\hat{H}_I(t) = -\frac{\Delta}{2} e^{-i\varphi(t)\hat{\sigma}_z} e^{i\varphi(t)\hat{\sigma}_x} e^{i\varphi(t)\hat{\sigma}_z} = -\frac{\Delta}{2} \begin{pmatrix} 0 & e^{-2i\varphi(t)} \\ e^{2i\varphi(t)} & 0 \end{pmatrix}.$$  

(3.11)

In the $\hat{\sigma}$-matrix form this Hamiltonian can be written as

$$\hat{H}_I(t) = -\frac{\Delta}{2} \left[ \hat{\sigma}_+ e^{-2i\varphi(t)} + \hat{\sigma}_- e^{2i\varphi(t)} \right].$$

(3.12)

For simplification of the Hamiltonian we use the following formulas with the Bessel functions

$$\exp \left[ i \frac{A}{\omega_1} \sin(\omega_1 t) \right] = \sum_{n_1} J_{n_1} \left( \frac{A}{\omega_0 + \delta} \right) e^{i(n_1\omega_0 + \delta)t},$$

(3.13)

where $J_{n}(x)$ is $n$-th order Bessel function of the first kind. In the result we can obtain

$$e^{2i\varphi(t)} = \sum_{n_1} \sum_{n_2} J_{n_1} \left( \frac{A}{\omega_0 + \delta} \right) J_{n_2} \left( \frac{A}{\omega_0 - \delta} \right) e^{i[\varepsilon_0 + (n_1 + n_2)\omega_0 + (n_1 - n_2)\delta]t}.$$  

(3.14)

We also add that $e^{-2i\varphi} = (e^{2i\varphi})^*$.

The resonance condition is formulated using the requirement that the oscillating terms in time have vanished. Thus, this condition is formulated for the central frequency $\omega_0$ and the electronic energy difference as $\varepsilon_0 - N\omega_0 =$
\[\Delta_N \ll \varepsilon_0, \text{ where } n_1 + n_2 = -N.\] In this approximation we obtain

\[e^{2i\varphi} = e^{i\Delta_N t} \sum_{n_1+n_2=-N} J_{n_1}(z_1) J_{n_2}(z_2)e^{i(n_1-n_2)\delta t}\]

\[= e^{i\Delta_N t} \sum_{n_1} J_{n_1}(z_1) J_{-N-n_1}(z_2)e^{i(2n_1+N)\delta t}\]

\[= e^{i(\Delta_N+N\delta)t-iN\pi} \sum_{n_1} J_{n_1}(z_1) J_{N+n_1}(z_2)e^{in_1\gamma}, \quad (3.15)\]

where \(z_1 = \frac{A}{\omega_0+\delta}, z_2 = \frac{A}{\omega_0-\delta}\) and \(\gamma = 2\delta t + \pi\). In the following we use the well-known formulas of summing the Bessel functions for the further transformation of the Hamiltonian. The result reads

\[e^{2i\varphi} = e^{i(\Delta_N+N\delta)t-iN\pi} J_N(w(t)) \left(\frac{z_2 - z_1 e^{-i\gamma}}{z_2 - z_1 e^{i\gamma}}\right)^\frac{N}{2}, \quad (3.16)\]

where \(w(t) = \left(z_1^2 + z_2^2 - 2z_1z_2 \cos(\gamma)\right)^{1/2}, |z_1 e^{\pm i\gamma}| < z_2\). We rewrite the exponent in the following form

\[e^{\pm2i\varphi(t)} = J_N(w(t))e^{\pm i\alpha(t)}, \quad (3.17)\]

introducing the function

\[\alpha(t) = (\Delta_N + N\delta)t - N\pi - \frac{iN}{2} \ln \left[\frac{z_2 - z_1 e^{-i\gamma}}{z_2 - z_1 e^{i\gamma}}\right]. \quad (3.18)\]

Equation (3.18) can be simplified easily if \(\delta \ll \omega_0\). Indeed, in this case \(z_1 \approx z_2\) when \(\delta \ll \omega_0\), and we can check that the logarithm in Eq. (3.18) is simplified
as
\[
\ln \left[ \frac{z_2 - z_1 e^{-i\gamma}}{z_2 - z_1 e^{i\gamma}} \right] \approx \ln \left( \frac{1 - e^{-i\gamma}}{1 - e^{i\gamma}} \right) \\
= \ln \left[ \frac{e^{-i\frac{\pi}{2}} (e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}})}{e^{i\frac{\pi}{2}} (e^{-i\frac{\pi}{2}} - e^{i\frac{\pi}{2}})} \right] = \ln \left( -e^{-i\gamma} \right) \\
= i(\pi - \gamma) = i(\pi - 2\delta t - \pi) = -2i\delta t.
\] (3.19)

Then, in the lowest approximation of \( \delta/\omega_0 \) we obtain \( \alpha(t) = \Delta_N t - N\pi \) and
\[
w(t) \approx 2\frac{A}{\omega_0} |\cos(\delta t)|.
\] (3.20)

In this approximation and for the case of exact resonance, \( \Delta_N = 0 \), the interaction Hamiltonian is written in the following form
\[
\hat{H}_I(t) = (-1)^{N+1} \frac{\Delta}{2} J_N(w(t)) \hat{\sigma}_x.
\] (3.21)

This Hamiltonian, describing the effects of time modulation on qubit dynamics, is nonstationary and \( T \) periodic, \( H_I(t + T) = H_I(t) \), with the period \( T = \pi/\delta \); thus, QESs and quasienergies can be introduced in this representation. The Hamiltonian is derived for the general case that involves one-quantum resonance process \( N = 1 \) as well as high-order processes with \( N > 1 \). Below we concentrate on consideration of two cases, \( N = 1 \) and 2, in detail.

### 3.3 Amplitudes of the tunneling and QES

The different regimes of qubit dynamics in the presence of a time-modulated field are formulated in the adiabatic and diabatic bases in analogy to the case of a monochromatic field [67, 116]. The diabatic basis states \( |\downarrow\rangle \) and \( |\uparrow\rangle \) are the eigenstates of the Hamiltonian Eq. (3.1), if \( \Delta \) and \( f(t) \) have vanished.
Let us consider the case $\varepsilon \gg \Delta$. We assume that states of a qubit are formed in the presence of a driving field and the tunneling process is described by transitions between these states. Then, in the lowest order of the perturbation theory on the basis of Eqs. (3.6) and (3.7) the tunneling amplitude in the transition $|\downarrow\rangle \rightarrow |\uparrow\rangle$ reads

$$A_{1 \rightarrow 2} = \langle \uparrow | \hat{H}_I(t) | \downarrow \rangle = \langle \uparrow (t) | \hat{H}_V(t) | \downarrow (t) \rangle,$$

where $|\uparrow (t)\rangle = \hat{U}(t)|\uparrow\rangle$, $|\downarrow (t)\rangle = \hat{U}(t)|\downarrow\rangle$ are the diabatic states in the $\hat{U}$ representation. In the limit of a weak driving we have $|\uparrow (t)\rangle = e^{-i\varepsilon t/2}|\uparrow\rangle$ and $|\downarrow (t)\rangle = e^{i\varepsilon t/2}|\downarrow\rangle$. For the amplitude we obtain

$$A_{1 \rightarrow 2} = (-1)^{N+1} \frac{\Delta}{2} J_N(w(t)). \quad (3.22)$$

This amplitude describes the tunneling transition in the presence of a time-modulated external field that shifts the energetic levels. It is interesting to compare this result with the analogous one for the case of an external monochromatic field. It is known that in the latter case the amplitude of the transition $|\downarrow\rangle \rightarrow |\uparrow\rangle$ with parameters satisfying the resonance does not depend on time intervals, while the amplitude Eq. (3.22) contains time-dependent periodic oscillations at the modulation frequency. In Fig. 3.1 and 3.2 we depict the corresponding probabilities of the tunneling transition in dependence on dimensionless time for two resonant conditions: $N = 1$ and $2$. As we see, the transition amplitudes are not constants and are periodic in time, while for the case of a one-monochromatic driving field these quantities have constant values.

In the case of a weak driving field $A \ll \varepsilon_0$ and $\omega_0 \sim \varepsilon_0$, we can use the following approximation for the Bessel function: $J_n(x) \sim \frac{x^n}{2^n n!}, x \ll 1$; therefore,

$$J_N(w(\tau)) = \frac{1}{N!} \left( \frac{A}{\omega_0} \right)^N \cos(\delta t)^N. \quad (3.23)$$
Thus, the amplitude of tunneling for a weak driving field is calculated as

$$A_{1\rightarrow 2} = (-1)^{N+1} \frac{\Delta}{2} \frac{1}{N!} \left( \frac{A}{\omega_0} \right)^N |\cos(\delta t)|^N. \quad (3.24)$$

This result is in accordance with the results of numerical calculations corresponding to first-order and second-order resonances presented in Fig. 3.1(a) and 3.2(a).

Below we turn to the general case of qubit dynamics considering the state of the full system in the $|\downarrow\rangle, |\uparrow\rangle$ basis as

$$|\Psi_U(t)\rangle = C_1(t)|\downarrow\rangle + C_2(t)|\uparrow\rangle. \quad (3.25)$$

In this case, the Schrödinger equation is reduced to two coupled first-order equations for the amplitudes in the following form

$$i\dot{C}_1(t) = -\frac{\Delta}{2} J_N(w(t))e^{-i\alpha(t)}C_2(t), \quad (3.26a)$$

$$i\dot{C}_2(t) = -\frac{\Delta}{2} J_N(w(t))e^{i\alpha(t)}C_1(t). \quad (3.26b)$$
The coefficients of these equations have a nontrivial dependence on time, nevertheless we demonstrate that for the resonance case, $\Delta_N = 0$, the solution of these equations can be found in a simple analytical form as follows:

\begin{align}
C_1(t) &= \cos(\gamma_N(t)), \quad (3.27a) \\
C_2(t) &= ie^{i\alpha} \sin(\gamma_N(t)), \quad (3.27b)
\end{align}

while the function $\gamma_N(t)$ is calculated from Eqs. (3.26a) and (3.26b) as

\[ \gamma_N(t) = \frac{2}{\Delta} \int_0^t J_N(w(\tau))d\tau \quad (3.28) \]

and $\alpha(t) = -N\pi$.

This solution is presented for the concrete initial conditions assuming that the system is initially in the lower state; therefore, $C_1(0) = 1$ and $C_2(0) = 0$. The populations of the initial and excited states (if the system was initially in
the lower state) as a function of time are then given by

\[ P_1(t) = |C_1(t)|^2 = \cos^2(\gamma_N(t)), \]  

(3.29a)

\[ P_2(t) = |C_2(t)|^2 = \sin^2(\gamma_N(t)). \]  

(3.29b)

To calculate these quantities further we need to analyze the function \( \gamma_N(t) \) that involves integration of a periodic function \( J_N(w(\tau)) \) with period \( T = \pi/\delta \). It is easy to represent the function \( \gamma_N(t) \) as

\[ \gamma_N(t) = \frac{\Delta}{2} J_N t + \Phi_N(t), \]  

(3.30)

where

\[ \overline{J_N} \equiv \overline{J_N(w(t))} = \frac{1}{T} \int_{t_0}^{t_0+T} J_N(w(\tau))d\tau, \]  

(3.31)

\( \Phi(t) \) is a periodic function defined for \( t \in [t_0, t_0 + T] \) as

\[ \Phi_N(t) = \frac{\Delta}{2} \int_{t_0}^{t} (J_N(w(\tau)) - \overline{J_N})d\tau, \]  

(3.32)

and for other \( t \in [0, \infty] \) through periodicity relation \( \Phi(t + T) = \Phi(t) \).

The above formulas allow us to introduce the QES of a qubit in a time-modulated driving field. Indeed, it is easy to check that the solution of Eq. (3.6) with periodic in time Hamiltonian (3.21) can be expressed in the adiabatic basis as

\[ |\Theta_{N,\pm}(t)\rangle = e^{\pm i(-1)^{N}\gamma_N(t)} |\varphi_{\pm}\rangle, \]  

(3.33)

where

\[ |\varphi_{\pm}\rangle = |\downarrow\rangle \pm |\uparrow\rangle. \]  

(3.34)

Then, by using the formula Eq. (3.30), these states can be presented in the form of a QES

\[ |\Theta_{N,\pm}(t)\rangle = e^{iE_N t} U_{N,\pm}(t) |\varphi_{\pm}\rangle, \]  

(3.35)
where

\[ U_{N,\pm}(t) = e^{\pm i(-1)^N \Phi_N(t)} \]

are periodic in time; \( U_{N,\pm}(t + T) = U_{N,\pm}(t) \), and \( E_N^\pm = \pm E_N \), where

\[ E_N = (-1)^N \frac{\Delta J_N}{2} \]

are the quasienergies. In the \( \Psi \) representation we obtain

\[ |\Psi_{\pm, N}\rangle = e^{iE_N^\pm t} U_{N,\pm}(t) \left( e^{-i\varphi(t)} |\downarrow\rangle \pm e^{i\varphi(t)} |\uparrow\rangle \right) \quad (3.38) \]

### 3.4 Quasienergies of the qubit in bichromatic field

In this section we study properties of the quasienergies. Note that some experiments recently realized in the field of superconducting Josephson qubits have been interpreted in terms of the probe absorption spectroscopy of the quasienergy levels (see, for example, \([80]\)). In this way, the frequencies of probe field absorption or amplification are determined by the matrix elements of transition between QESs. We briefly discuss this problem, considering the transition \( |\Theta_{N, +}\rangle \rightarrow |\Theta_{N, -}\rangle \) between quasienergetic states \( \Theta_{N, \pm}(t) \) due to a weak interaction of the system with a probe field. Such an interaction with a probe field at the frequency \( \omega_p \) can be added as weak perturbation term \( \lambda E_p \cos(\omega_p t) \sigma_z \) in the Hamiltonian (3.1). Thus, the matrix element of this transition is calculated as

\[ \langle \Theta_{N, -} | \sigma_z | \Theta_{N, +} \rangle = 2 \sum_{n, m} J_n \left( \frac{A}{\omega_0} \right) J_{m-n} \left( \frac{A}{\omega_0} \right) \exp(i\Omega_{m,n} t), \quad (3.39) \]

where \( m = 0, \pm 1, \pm 2, \ldots; n = 0, \pm 1, \pm 2, \ldots \); and \( \Omega_{m,n} = \varepsilon_0 + m\omega_0 + (E^+ - E^-) + (2n-m)\delta \) are the frequencies of spectral lines corresponding to the absorption (for \( \omega_p = \Omega_{m,n} > 0 \)) and the amplification (for \( \omega_p = \Omega_{m,n} < 0 \)) of a probe field. As we can see, the spectral lines separated by the central frequency and
modulation harmonics and contain a field-dependent Stark shift due to the
input of the quasienergies. In Sec. 3.5 we demonstrate that the quasienergies
in Eq. (3.37) also play an essential role in occupation populations of states.

As we see, the sum of two quasienergies obeys the relation \( E_N^+ + E_N^- = 0 \).
This result is in accordance with the exact result taking place for a two-level
atom in a monochromatic field. According to \[69\] the sum of two quasiener-
gies equals the sum of atomic energetic levels, that is zero for the case of
the truncated Hamiltonian Eq. (3.1), in which the half of the sum of qubit
energetic levels has been omitted.

The difference between quasienergies reads \( E_N^+ - E_N^- = 2E_N \) in this case.
In this representation quasienergies contain only a field-dependent part and
equal zero in the limit of small driving. Dependences of the quasienergy
\( E_N^{\pm} \) on the parameter \( A/\omega_0 \) as a function \( E_N^{\pm} = E_N^{\pm}(A/\omega_0) \) for two types of
resonances, for the first order as well as for the second order, are shown in
Fig. 3.3. As we demonstrate, the quasienergies \( E_1 \) and \( E_2 \) for both types of

\[ \text{Figure 3.3: Quasienergy for } (N = 1) \text{ first-order resonance (solid curve);}
\]
\[ (N = 2) \text{ second-order resonance (dashed curve).} \]

resonances have zeros for the definite values of \( A/\omega_0 \). The lower zeros are at
\( A/\omega_0 = 3.13, 6.3, \text{ and } 9.45 \) for \( N = 1 \) and are at \( A/\omega_0 = 3.8, 7.05, \text{ and } 10.2 \) for
\( N = 2 \). Below, we also analyze the quasienergies for the regime of a weak
external field.
3.4.1 Quasienergies and phase function at the regime of weak driving

In this subsection we derive approximative analytical results for the quasienergies and the phase function Eq. (3.30) using the formula Eq. (3.23), which describes the weak driving limit. Integration of the formula (3.31) leads to

\[
\mathcal{J}_N = \frac{\delta}{\pi} \int_0^{\pi/\delta} J_N(w(\tau))d\tau = \frac{F_{[0;\pi],N}(\pi/\delta)}{N!\pi} \left( \frac{A}{\omega_0} \right)^N
\]

\[
= \frac{2\Gamma \left( \frac{3+N}{2} \right) + (1+N)\Gamma \left( \frac{1+N}{2} \right)}{2\sqrt{\pi}N!(1+N)\Gamma \left( \frac{3+N}{2} \right)} \left( \frac{A}{\omega_0} \right)^N.
\] (3.40)

In this formula, we introduce a function, which also determines the periodic part of the phase function Eq. (3.32),

\[
F_{[0;\pi],N}(t) = \frac{\sqrt{\pi}}{2} \frac{\Gamma \left( \frac{1+N}{2} \right)}{\Gamma \left( 1 + \frac{N}{2} \right)} - \frac{2F_1 \left( \frac{1}{2}, \frac{1+N}{2}; \frac{3+N}{2}; \cos^2(\delta t) \right)}{1+N} \cos(\delta t)\cos(\delta t)N
\]

\] (3.41)

which is defined in \([0; \pi/\delta]\). Here, \(2F_1(a, b; c; z)\) is a hypergeometric function. The final result for the quasienergy reads as follows:

\[
E_N = (-1)^N \frac{\Delta}{2} \frac{2\Gamma \left( \frac{3+N}{2} \right) + (1+N)\Gamma \left( \frac{1+N}{2} \right)}{2\sqrt{\pi}N!(1+N)\Gamma \left( \frac{3+N}{2} \right)} \left( \frac{A}{\omega_0} \right)^N,
\] (3.42)

while the periodic part of the phase function is calculated as

\[
\Phi_N(t) = \frac{\Delta}{2} \int_0^t (J_N(w(\tau)) - \mathcal{J}_N)d\tau
\]

\[
= \frac{\Delta}{\delta} \frac{1}{2N!} \left( \frac{A}{\omega_0} \right)^N \left[ F_{[0;\pi],N}(t) - F_{[0;\pi],N}(\pi/\delta)\delta t \right].
\] (3.43)

Note, that the results of this section on the quasienergetic states of the qubit in a time-modulated driving field are essentially different from the
analogous results for the states obtained for a two-level atomic system driven by a bichromatic field with the Hamiltonian Eq. (3.5) [76, 98, 100–102]. We demonstrate this point below.

3.4.2 System with time-dependent component along $x$ axis

In this subsection we briefly discuss the system with the Hamiltonian Eq. (3.5), which is typical for problems that involve an atom in a bichromatic laser field. Our goal is to show the differences of the behaviors for the cases of superconducting qubits [see Hamiltonian Eq. (3.1 with the time-dependent component along $z$ axis] and two-level atomic system [see Hamiltonian Eq. (3.5) with the time-dependent component along $x$ axis] in bichromatic field.

We now take the system described by the Hamiltonian Eq. (3.5) in new denotations that are more standard in this area:

$$
\hat{H} = -\frac{\Delta E}{2} \hat{\sigma}_z + V \cos(\omega_0 t) \cos(\delta t) \hat{\sigma}_x.
$$

(3.44)

We make a transformation to a rotating frame $|\Psi(t)\rangle = \hat{W}(t)|\Psi_W(t)\rangle$, where

$$
\hat{W}(t) = \exp \left( \frac{i}{2} \Delta E t \hat{\sigma}_z \right).
$$

(3.45)

For this system we can formulate only a one-quantum condition of the resonance, $\Delta E = \omega_0$, in contrast to the system with the time-dependent component along the $z$ axis in which multiquantum resonances take place. In the resonance approximation we obtain

$$
i \frac{\partial}{\partial t} |\Psi_W(t)\rangle = V \cos(\delta t) \hat{\sigma}_x |\Psi_W(t)\rangle.
$$

(3.46)
The solution of this equation in the adiabatic bases can be obtained as

$$|\Phi_{\pm}(t)\rangle = \exp\left(\frac{iV}{\delta} \sin(\delta t)\right) |\varphi_{\pm}\rangle.$$ (3.47)

Comparing the results of Eqs. (3.35) and (3.47) we conclude that the quasienergies corresponding to QES Eq. (3.47) are equal to zero for all ranges of the parameters in the rotating wave approximation in contrast to the results of Eqs. (3.35), (3.37). Besides this, the periodic wave function \(\exp(i\frac{V}{\delta} \sin(\delta t))\) strongly differs from the periodic wave function \(U_{N,\pm}(t)\) that corresponds to QES Eq. (3.35). This situation is displayed also in the frequencies of spectral lines corresponding to the transitions between QES \(|\Phi_{\pm}(t)\rangle\). Indeed, it is easy to realize that these frequencies are at \(\omega_p = \omega_0 + n\delta, n = 0, \pm 1, \pm 2, \ldots\) and do not involve field-dependent shifts of energetic levels. This effect is in accordance with calculation of the spectrum of resonance fluorescence and Autler-Townes splitting in a bichromatic field [100, 102].

At the end of this subsection, for completeness, we present the QES in the \(\Psi\) representation:

$$|\Psi_{+}(t)\rangle = \hat{W}(t)|\Phi_{+}(t)\rangle = \frac{1}{\sqrt{2}} \exp\left(i\frac{V}{\delta} \sin(\delta t)\right) \left(e^{-i\frac{\Delta E}{2}} |\downarrow\rangle + e^{i\frac{\Delta E}{2}} |\uparrow\rangle\right),$$ (3.48)

$$|\Psi_{-}(t)\rangle = \hat{W}(t)|\Phi_{-}(t)\rangle = \frac{1}{\sqrt{2}} \exp\left(-i\frac{V}{\delta} \sin(\delta t)\right) \left(e^{-i\frac{\Delta E}{2}} |\downarrow\rangle - e^{i\frac{\Delta E}{2}} |\uparrow\rangle\right).$$ (3.49)

It should be noted that the Floquet basis derived here for a qubit in a bichromatic field is useful for studying the Rabi oscillation physics as well as for writing the master equation governing the dynamics of the reduced density matrix of a driven system, which is in contact with an external environment.
Note that Hamiltonians Eq. (3.1) and (3.44) are the particular cases of a more general Hamiltonian:

$$\hat{H}(t) = -\frac{1}{2}B_z(t)\hat{\sigma}_z - \frac{1}{2}B_x(t)\hat{\sigma}_x,$$

(3.50)

which can be realized on a properly designed superconducting circuit. In particular, a simple design of the charge qubit with tunable effective Josephson coupling can be shown schematically (see Fig. 3.4) as with time-dependent coefficients $B_x(t)$ and $B_z(t)$ that allow complete control of the system through the gate voltage $V_g$ and the external magnetic flux $\Phi_x$ (see, for example [50]). The relations between these quantities are expressed as

$$B_x(t) = E_J(\Phi_x(t)) = 2E_J^0 \cos \left( \frac{\Phi_x(t)}{\Phi_0} \right),$$

(3.51)

$$B_z(t) = \delta E_{ch}(V_g(t)) = 4E_C \left( 1 - \frac{1}{e}C_g V_g(t) \right),$$

(3.52)

where $E_C$ is the single-electron charging energy, $E_J^0$ is the Josephson coupling energy, and $C_g$ is the gate capacitor. Practical realizations of an analogous scheme have been done in a series of papers (see, for example, [172, 173]). We believe that the schemes considered here with a time-dependent component along the $z$ or $x$ axis driven by bichromatic external fields can be
constructed in the same way (see [68], in which multisideband components of qubit energy splitting are observed).

### 3.5 Aperiodic and periodic Rabi oscillations

In this section, time-dependent populations of states are investigated for various regimes. We investigate dynamics of the driven qubit in a time domain for various resonance conditions. Thus, we consider the occupation probability as a function of time in dimensionless units, assuming that the system was initially in the state $\ket{\downarrow}$ and the Rabi frequency is given by

$$\Omega_N(t) = \dot{\gamma}_N(t).$$  \hfill (3.53)

According to the formulas Eqs. (3.27) and (3.29) this dynamics is determined by the function $\gamma_N(t)$, that involves both the quasienergy and the periodic function $\Phi_N(t)$ with period $T = \pi/\delta$. To present this statement in a clear form we rewrite the formula Eq. (3.30) as

$$\gamma_N(t) = (-1)^N E_N t + \Phi_N(t).$$  \hfill (3.54)

The function $\gamma_N(t)$ is an increasing function in time but it grows also periodically due to its "linear+periodic" structure. Therefore, the dynamics of populations Eq. (3.29) seems to be aperiodic in time. Indeed, the typical results for the phase function as well as the populations are depicted in Figs. 3.5, 3.6 and 3.7. The dynamics of populations for the case of a weak external field is shown in Figs. 3.5 for two resonance regimes. In Fig. 3.5(a) we compare two curves of the occupation probabilities for $N = 1$ (solid curve) and for $N = 2$ (dashed curve). We can see here fast oscillations of the population for the regime $N = 1$ and slow oscillations for the case of $N = 2$ (for consideration in details, see the curve corresponding to the case $N = 2$ for large time intervals in Fig. 3.5(b)). The results for the second-order resonance regime are
also demonstrated in Fig. 3.5(c) for the other parameter $\Delta/\delta$. Analyzing these results, we note that dynamics of populations strongly depends on the value of the ratio $\Delta/\delta$. It can be seen from the formulas Eqs. (3.40) and (3.43) that population behavior shown in Fig. 3.5(b) for $N = 2$ is mainly governed by the linear in time term in the phase function Eq. (3.43); thus, we can see that the dynamics looks like cosinusoidal oscillations. The periodic in time part $\Phi_N(t)$ only slightly modulated these oscillations. This part of the phase function increases with increasing the parameter $\Delta/\delta$ that leads to increasing the role of periodic modulations giving rise to a nontrivial time dependence of occupation probability [see, Fig. 3.5(c) for the case $N = 2$].

The typical examples of occupation probabilities corresponding to the large-amplitude regime are depicted in Figs. 3.6 and 3.7. In order to illustrate the role of phase function in the development of aperiodic dynamics
we also show here the time dependence of this function.

![Figure 3.6](image)

**Figure 3.6:** (a) Phase-function $\gamma_N(t)$ and (b) population probability for the first-order resonance condition ($N = 1$). The parameters are: $\Delta/\delta = 12$, $A/\omega_0 = 1$.

It is obvious that the dynamics of populations can be periodic if the quasienergy becomes equal to zero at definite values of the parameter $A/\omega_0$ (see, Fig. 3.3). However, as it can be seen, this situation also takes place for the other wide ranges of the parameters, if the shift of the phase function during $m$ periods $T = \pi/\delta$

$$\gamma(t + mT) = (-1)^N E_N(t + mT) + \Phi_N(t)$$

$$= \gamma(t) + (-1)^N \frac{E_N \pi}{\delta} m$$  \hspace{1cm} (3.55)

becomes equal to $n\pi$, that is the period of square $\cos(x)$ or $\sin(x)$ in the formulas Eq. (3.29). Such a consideration leads to the following formula:

$$m\delta = n|E_N|,$$  \hspace{1cm} (3.56)

where $m$ and $n$ are positive integers. The physical means of this formula is very simple. The population of the states depends on $\gamma_N(t)$ as a square of the cosine, for example $P_1 = \cos^2[\gamma_N(t)]$. Thus, if during $m$ periods its growth
is equal to any period of \( \cos^2(x) \), which can be written as \( n\pi \), the population will repeat its behavior.

Thus, the populations could be made periodic by choosing the values of parameters \( \delta/\Delta \) and \( A/\omega_0 \) to satisfying the following condition:

\[
\frac{\delta}{\Delta} = \frac{n}{m} \frac{1}{2\pi} \int_{0}^{\pi} J_N \left( \frac{2A}{\omega_0} |\cos(\tau)| \right) d\tau,
\] 

(3.57)
that follows from the formulas Eqs. (3.37) and (3.56). For the case of a weak driving field this condition is simplified and reads

$$\frac{1}{N!} \frac{\Delta}{2\delta} \left( \frac{A}{\omega_0} \right)^N \sqrt{\pi} \frac{\Gamma\left(\frac{1+N}{2}\right)}{\Gamma\left(1 + \frac{N}{2}\right)} = \pi \frac{m}{n}. \quad (3.58)$$

The typical results for Rabi oscillations with regular, periodic dynamics are depicted in Fig. 3.8 for the $N = 2$ resonance condition. Here, the parameters $A/\omega_0$ and two used parameters, $\Delta/\delta = 401$ [see Figs. 3.8(a)] and $\Delta/\delta = 31$ [see Figs. 3.8(b)], satisfy the periodicity condition Eq. (3.57). We compare the results shown in Fig. 3.8(a) with the result depicted in Fig. 3.5(c). Both results are obtained for the second-order resonance condition and for the same parameter $A/\omega_0 = 10^{-1}$; however, using the parameter $\Delta/\delta$ satisfying the condition of periodicity Eq. (3.57) in Fig. 3.8(a) leads to the periodic dynamics of the populations. These regimes in which quantum dynamics of occupation probabilities becomes periodically regular can be useful, for example, in applications where one is dealing with logic operations on qubits.
Chapter 4

DYNAMICS OF QUANTUM TRANSITIONS IN MAGNUS REPRESENTATION

4.1 Dynamics of a two-level quantum system in the Magnus representation on the Bloch sphere

Quantum information technology with the qubits based on trapped ions, atoms in resonator, quantum wells and superconducting systems is usually realized on the two-level systems under the influence of optical or microwave pulses. However, the general analytical solution of dynamics of this system is not found until now, in particular, in calculations of quantum transitions probabilities between the qubit states without various approximations. The known approximation is the resonance one (approximation of rotating wave) in which the effects of oscillating terms are neglected [86–88]. Recently the strong coupling regimes of two-level quantum system with the radiation field has been considered, where the effects beyond the resonance approximation become important [89, 90]. The investigations of phenomena
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beyond the framework of rotating wave approximation are cited in a series of works [91–97]. The resonant interaction of atom with the bichromatic field is considered in [75, 76, 98–103], and in the case of amplitude modulation in [104]. The resonant effects of atom interaction with the laser field are fully investigated in the nonlinear [105, 106] and atom [107] optics. In addition to the numerous works in this area, in the present work the approach is given which enables one to generalize the investigation of two-level atom system beyond the limits of the resonance approximation. It is known that the investigation of the perturbed two-level system is simplified when the Bloch variables are used for the vector of state [108, 109]. In the approach supposed below, the matrix of temporal evolution of quantum state in the Magnus formulation [110] described in the terms of Bloch variables is used. Within the frames of such an approach one can obtain the general analytical expressions for transition probabilities between the states of an atom beyond the resonant approximation.

4.1.1 Matrix evolution in Magnus representation

We consider the standard model of two-level atom interacting with the perturbation field. The Hamiltonian of the system is written in the following form:

$$\hat{H}(t) = \frac{\varepsilon_0}{2} \hat{\sigma}_z + V g(t) \hat{\sigma}_x, \quad (4.1)$$

where $\varepsilon_0$ is the frequency of atomic transition, $V$ describes the coupling of two-level atom with the field, $\hat{\sigma}_x$ and $\hat{\sigma}_y$ are the Pauli matrices. In this case the state of the system is defined as

$$|\Psi(t)\rangle = C_1(t)|1\rangle + C_2(t)|2\rangle, \quad (4.2)$$
where amplitudes $C_1(t)$ and $C_2(t)$ satisfy the following equations:

\[ i\dot{C}_1(t) = \frac{\varepsilon}{2}C_1(t) + Vg(t)C_2(t), \quad (4.3) \]
\[ i\dot{C}_2(t) = -\frac{\varepsilon}{2}C_2(t) + Vg(t)C_1(t), \quad (4.4) \]

Introducing notations

\[ C_1(t) = \tilde{C}_1e^{-i\varepsilon_0 t}/2, \quad (4.5) \]
\[ C_2(t) = \tilde{C}_2e^{i\varepsilon_0 t}/2, \quad (4.6) \]

these equations may be written in the matrix form

\[
\begin{pmatrix}
\dot{\tilde{C}}_1(t + dt) \\
\dot{\tilde{C}}_2(t + dt)
\end{pmatrix}
= \begin{pmatrix}
1 & -iVg(t)e^{i\varepsilon_0 t}dt \\
-iVg(t)e^{-i\varepsilon_0 t}dt & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{C}_1(t) \\
\tilde{C}_2(t)
\end{pmatrix}, \quad (4.7)
\]

and then represent the temporal evolution of the system with the use of evolution operator in the form of the Bloch variables

\[
\begin{pmatrix}
\tilde{C}_1(t + dt) \\
\tilde{C}_2(t + dt)
\end{pmatrix}
= \hat{U}(t, t + dt)
\begin{pmatrix}
\tilde{C}_1(t) \\
\tilde{C}_2(t)
\end{pmatrix}. \quad (4.8)
\]

Where the evolution operator is described as the rotation operator on the Bloch sphere

\[
\hat{U}(t, t + dt) = \hat{I} - i\frac{d\theta(t)}{2} \left(\tilde{n}\hat{\sigma}\right), \quad (4.9)
\]

and in new notations equations (4.8) are described as

\[
\begin{pmatrix}
\dot{\tilde{C}}_1(t + dt) \\
\dot{\tilde{C}}_2(t + dt)
\end{pmatrix}
= \begin{pmatrix}
1 - in_z\frac{d\theta}{2} & (-in_x - n_y)\frac{d\theta}{2} \\
(-in_x + n_y)\frac{d\theta}{2} & 1 - in_z\frac{d\theta}{2}
\end{pmatrix}
\begin{pmatrix}
\tilde{C}_1(t) \\
\tilde{C}_2(t)
\end{pmatrix}. \quad (4.10)
\]
where $d\theta = 2Vg(t)dt$ and the components of vector $n_x$, $n_y$ and $n_z$ are equal to:

\[ n_x = \cos(\varepsilon_0 t), \quad n_y = -\sin(\varepsilon_0 t), \quad n_z = 0. \]  

(4.11)

Thus, the state of the system may be represented as

\[ |\Psi(t + dt)\rangle = \hat{U}(t, t + dt)|\Psi(t)\rangle, \]  

(4.12)

and, following the standard approach of the perturbations theory, the temporal evolution of quantum state, according to the Dyson expansion formula, may be represented as

\[ |\Psi(t)\rangle = \hat{U}(t, 0)|\Psi(0)\rangle = T \exp \left( -\frac{i}{2} \int_0^t \left( \vec{n} \hat{\vec{\sigma}} \right) d\theta \right) |\Psi(0)\rangle, \]  

(4.13)

where $T$ denotes the chronological ordering of operators. In the present work, the alternative expansion of evolution operator in the Magnus form is used, where the operation of chronological ordering is absent

\[ \hat{U}(t, 0) = \exp \left( \sum_{k=1}^{\infty} \hat{\Omega}_k(t) \right). \]  

(4.14)

Here, the first three terms of Magnus expansion have the form

\begin{align*}
\hat{\Omega}_1(t) &= -iV \int_0^t \hat{h}(t_1)dt_1, \\
\hat{\Omega}_2(t) &= -\frac{V^2}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 \left[ \hat{h}(t_1, \hat{h}(t_2)) \right], \\
\hat{\Omega}_3(t) &= i\frac{V^3}{6} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \left( \left[ \hat{h}(t_1, \hat{h}(t_2, \hat{h}(t_3))) \right] + \left[ \hat{h}(t_3, \hat{h}(t_2, \hat{h}(t_1))) \right] \right).
\end{align*}  

(4.15)
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and for the case of Hamiltonian (4.1) they are expressed by the Pauli matrices in the following form:

\[
\hat{h}(t) = g(t) [\cos(\varepsilon_0 t)\hat{\sigma}_x - \sin(\varepsilon_0 t)\hat{\sigma}_y]. \tag{4.16}
\]

The calculations result in expressions

\[
\hat{\Omega}_1(t) = -iV (F_c(t)\hat{\sigma}_x - F_s(t)\hat{\sigma}_y),
\]

\[
\hat{\Omega}_2(t) = iV^2 \left( F_c(t)F_s(t) - 2 \int_0^t F_c(t')dF_s(t') \right) \hat{\sigma}_z, \tag{4.17}
\]

where

\[
F_c(t) = \int_0^t g(t') \cos(\varepsilon_0 t')dt',
\]

\[
F_s(t) = \int_0^t g(t') \sin(\varepsilon_0 t')dt'. \tag{4.18}
\]

Easy to see that each term of Magnus expansion is expressed in terms of Pauli matrices. Within the frameworks of such an approach the evolution operator is represented in a simple form for all of the terms of Magnus expansion (4.14)

\[
\hat{U}(t, 0) = \exp(i\tilde{A}(t)\tilde{\sigma}) = \hat{I} \cos(A(t)) + i \left( \tilde{\rho}\hat{\sigma} \right) \sin(A(t)), \tag{4.19}
\]

where \( A(t) = |\tilde{A}(t)| \) and

\[
\tilde{\rho} = \frac{\tilde{A}(t)}{A(t)}. \tag{4.20}
\]

In the matrix form, for the evolution operator we obtain

\[
\hat{U}(t, 0) = \begin{pmatrix}
\cos(A(t)) + i\rho_z \sin(A(t)) & (i\rho_x + \rho_y) \sin(A(t)) \\
(i\rho_x - \rho_y) \sin(A(t)) & \cos(A(t)) - i\rho_z \sin(A(t))
\end{pmatrix}. \tag{4.21}
\]
This leads to the following expressions for the amplitudes of the vector of state:

\[
\begin{align*}
C_1(t) &= e^{-\frac{i\epsilon_0 t}{2}} [\cos(A(t)) + i\rho_z \sin(A(t))] C_1(0) + e^{-\frac{i\epsilon_0 t}{2}} (i\rho_x + \rho_y) \sin(A(t)) C_2(0), \\
C_2(t) &= e^{\frac{i\epsilon_0 t}{2}} (i\rho_x - \rho_y) \sin(A(t)) C_1(0) - e^{\frac{i\epsilon_0 t}{2}} [\cos(A(t)) - i\rho_z \sin(A(t))] C_2(0)
\end{align*}
\] (4.22)

Thus, the problem reduces to the calculation of function \(A(t)\) in each order of the Magnus expansion. Up to the second order, the expansion results in the following formulas:

\[
\begin{align*}
A_x(t) &= -VF_c(t), \\
A_y(t) &= -VF_s(t), \\
A_z(t) &= -V^2 \left( F_c(t) F_s(t) - 2 \int_0^t F_c(t') dF_s(t') \right).
\end{align*}
\] (4.23)

### 4.1.2 Transition probabilities and Raby oscillations

Consider the probability of excitation of the two-level atom if the system is in the ground state \(C_2(0) = 0\) at the shutting of interaction. From the set of equations (4.22) we obtain

\[
P_2(t) = |C_2(t)|^2 = (1 - \rho^2) \sin^2(A(t)).
\] (4.24)

If the initial state is the superposition of \(C_1 \neq 0\) and \(C_2(0) \neq 0\) states, for the probability of excitation we obtain

\[
P_2(t) = [\rho_y \sin(A(t)) C_1(0) + \cos(A(t)) C_2(0)]^2 + [\rho_x C_1(0) + \rho_z C_2(0)]^2 \sin^2(A(t)).
\] (4.25)

These results have the general character and ones describe the transition probabilities of the system from the initial state for the arbitrary interaction with the external field.
Consider the important case of system interaction with an external field in the form of pulse \( g(t) = f(t) \sin(\omega t) \) with an amplitude \( f(t) \) and the carrying frequency \( \omega \). If the \( \omega = \varepsilon \) resonance is present, the expressions (4.23) simplifies considerably. Easy to show that in this case \( A_x(t) = 0 \), \( A_y(t) = \frac{1}{2} \int_0^t f(t') dt' \) and \( A_z(t) \). Thus, the correlator of operators in (4.15) is vanishing and only the terms of the lowest, first order of Magnus expansion are not vanishing. The value of \( \rho_z \) is equal to zero, \( \rho_z = 0 \), therefore, we obtain the well-known result for the probability (4.24)

\[
P_2(t) = |C_2(t)|^2 = \sin^2 \left( \frac{1}{2} \int_0^t f(t') dt' \right).
\] (4.26)

As for the probability (4.26), one obtains the following form in the resonant approximation

\[
P_2(t) = \left[ \sin \left( \frac{1}{2} \int_0^t f(t') dt' \right) C_1(0) + \cos \left( \frac{1}{2} \int_0^t f(t') dt' \right) C_2(0) \right]^2. \tag{4.27}
\]

Formulas (4.25) and (4.26) are also convenient for calculations beyond the frames of the resonant approximation. In particular, the iteration procedure may be formulated to calculate the corrections to the resonant approximation by the series expansion parameter, which is the ratio of matrix element of transition to the frequency of the perturbation field. Below we present the results for two cases of two-level atom interaction with the monochromatic field and the Gaussian pulses.

First consider the case of non-resonant interaction of atom with the monochromatic field \( g(t) = \sin(\omega t) \), where the frequency \( \omega \) is differing from the atom transition frequency \( \varepsilon_0 \). In the ground state of an atom the transition probability \( P_2(t) \) is calculated according to the formula (4.24) with the quantities \( A(t) \) (4.23) in the second order of the Magnus expansion. The temporal dependence of the transition probability in dimensionless units is shown in Fig.4.1 for two frequency values. Easy to see that the probability to find
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**Figure 4.1:** Temporal dependence of probability $P_2(t)$ to find the atom in the state $|2\rangle$ by interaction with the external monochromatic field. Curve 1: $V = 0.375, \omega = 1.5$ in $\varepsilon_0$ units, curve 2: $V = 0.375, \omega = 1.2$.

**Figure 4.2:** Temporal dependence of probability $P_2(t)$ to find the atom in the state $|2\rangle$ by interaction with the external Gaussian pulse with the width $T = 8$ and the centrum $t_0 = 4$ in $1/\varepsilon_0$ units. Curve 1: $V = 0.375, \omega = 1.2$ in $\varepsilon_0$ units, curve 2: $V = 0.375, \omega = 1.5$, curve 3: $V = 0.5, \omega = 1.2$.

the system in the excited state by interaction with the field at the frequency $\omega = 1.2\varepsilon_0$ reaches the value 0.5, several times higher than the maximum of the population of this state with the field at the frequency $\omega = 1.5\varepsilon_0$.

Consider now the case of non-resonant interaction of atom with the pulse the envelope amplitude of which is the Gauss function

$$g(t) = \exp \left( -\frac{(t - t_0)^2}{T^2} \right) \sin(\omega t).$$  (4.28)
The calculations are similar to the preceding case with the formulas (4.22) - (4.25). The results of calculations are shown in Fig. 4.2. The influence of Gaussian pulses at the frequencies $\omega = 1.2\varepsilon_0$ and $\omega = 1.5\varepsilon_0$, but with equal amplitudes is compared with the pulse at the frequency $\omega = 1.2\varepsilon_0$ with the greater amplitude $V = 0.5$.

The temporal dependence of probability $P_2(t)$ to find the atom in the state $|2\rangle$, for the different modes which is shown in Fig. 4.2, illustrates a predictable result. The Gaussian pulse with the frequency close to the resonant frequency and interacting with the qubit results in the inversion of population. However, the higher pulse amplitude improves this result.

### 4.2 Multiphoton resonant manipulation of qubits by train of pulses

Interactions of quantum systems with pulse train are of great interest for many scientific and technological applications, such as qubit manipulation or quantum computing, secure communications and ultra-precise measurements. This direction is similar to well known method of nuclear magnetic resonance for the radio-frequency control and manipulation of spin systems, and has also been applied in atomic physics (see, for example: [111], [112]), in development of ultrafast quantum gates with trapped ions [113], in magnetometry with solid state quantum sensors [114], molecular spectroscopy [115] and driven quantum tunneling associated with a wide variety of interesting phenomena and effects [116].

Recently, manipulation and control of the states of single superconducting qubit by resonant microwave pulses have been extensively studied [55, 57, 125–127]. Particularly, the dynamics of superconducting qubit driven by external field with time-modulated amplitude and the phenomenon of Rabi oscillations have been considered [104] close to the experimental scheme on
the frequency-modulated transmon qubit [68]. Superconducting qubits usually have short coherence time, therefore to decrease the time for performing gate operations a large-amplitude external fields should be applied. The dynamics of a qubit driven by large-amplitude external fields in the case of driving around the region of avoided level crossing has been also studied (see, [66, 67] for reviews). Recent experimental investigations of the dynamics of strongly driven superconducting qubit are presented in [174].

In the present paper we present a systematic approach for investigation of tunneling quantum dynamics of two-level systems, particularly, superconducting qubits interacting with a single pulse as well as with a train of pulses. In this way, we analyse the Rabi model on base of the Magnus QED formalism in Furry representation for calculation of time-evolution operator. The Magnus formalism [110, 128] leads to expansion of the time-evolution operator in the exponential form that contains all terms of perturbation theory on coupling constant without Dyson time-ordering in QED. The advantages of this approach for periodically driven systems recently have been discussed in details [129]. The Furry picture (see, for example [130]) gives a simple tool for describing processes in an external field. Thus, this approach allows us to obtain general explicit solutions for single-qubit populations as a function of time-dependent Rabi frequency and allows to apply the rotating-wave approximation (RWA) as well as to consider naturally the system beyond RWA. Indeed, by using the Magnus formalism in the Furry representation we can formulate an effective truncation procedure for Magnus series: in RWA the operator of time-evolution is only determined by the first term of Magnus series and effects beyond RWA can be calculated as corrections to the resonant part through the high-order Magnus terms.

In this way, we demonstrate the results for a wide range of frequencies of driving field corresponding to multiphoton resonant excitation regimes of qubit for various pulse regimes. The concrete calculations will be done for the train of pulses with Gaussian envelopes and with different phases. The
duration of pulses, time intervals between them and the relative phases of pulses are the free control parameters.

Another interesting regime occurs when the pulse train consists of identical pulses and hence displays the periodicity in time. We demonstrate that quasienergies (QE) and quasienergetic states (QES) (or so-called the Floquet states) for the qubit driven by train of identical pulses are realized in multiphoton resonant regimes. We also observe that regular oscillations of the state population is realized if the factor of the quasienergy and the period becomes equal to $m\pi$, where $m = 1, 2, \ldots$.

### 4.2.1 Magnus approach in Furry picture

In this section, we derive general expressions for populations of a single qubit with applications for various multiphoton resonant excitation regimes including: single-pulse excitation and two-pulse excitation with phase shift between pulse envelopes being controlling parameter.

At first, we consider the qubit driven by a field of pulses with arbitrary envelopes on base of the Hamiltonian

$$
\hat{H}(t) = -g(t)\hat{\sigma}_z + \Delta \hat{\sigma}_x,
$$

where time-dependent interaction takes place along the $z$ axis with the coupling constant

$$
g(t) = \frac{1}{2}\varepsilon_0 + A(t)\cos(\omega t + \theta).
$$

Here, $\Delta$ is the coupling strength or the tunneling amplitude between the basis states, $\varepsilon_0$ is the electronic energy difference between the ground and excited states of the qubit in the absence of the driving field and the coupling strength. $g(t)$ includes the interaction between the external time-dependent field and the two-level system, and $A(t)$ is the amplitude of external field,
\(\hat{\sigma}_x, \hat{\sigma}_y\) and \(\hat{\sigma}_z\) are three ordinary Pauli matrices. The Hamiltonian (4.29) describes various physical systems including quantum two-level systems with nonlinear tunneling mechanism driven by external field [116] and also superconducting qubit or \(JJ\) artificial atoms with \(\Delta\) being the Josephson coupling energy. Up to now, numerous proposals and demonstrations of quantum phenomena in superconducting qubit have been demonstrated in the fields of atomic physics and quantum optics (see, for example review papers [52, 175]).

We start with time-evolution equation of the system

\[
i \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H}(t) |\Psi(t)\rangle
\]

(4.31)
in the Furry picture \(|\Psi(t)\rangle = \hat{U} |\Phi(t)\rangle\) where the operator \(\hat{U}\) is given as

\[
\hat{U} = \exp \left( i \int_0^t g(t') dt' \cdot \hat{\sigma}_z \right)
\]

(4.32)
and involves explicit interaction of the qubit with external field. Then time-evolution equation of the system can be rewritten in the following form

\[
i \frac{\partial}{\partial t} |\Phi(t)\rangle = \hat{h}(t) |\Phi(t)\rangle,
\]

(4.33)
where

\[
\hat{h} = \Delta \left[ e^{-2i\Lambda(t)} \hat{\sigma}_+ + e^{2i\Lambda(t)} \hat{\sigma}_- \right],
\]

(4.34)
\(\hat{\sigma}_+ = (\hat{\sigma}_x + i\hat{\sigma}_y)/2, \hat{\sigma}_- = (\hat{\sigma}_x - i\hat{\sigma}_y)/2\) and

\[
\Lambda(t) = \int_0^t g(t') dt'.
\]

(4.35)

Following a standard perturbation approach time-evolution of quantum states \(|\Phi(t)\rangle = S(t, 0) |\Phi(0)\rangle\), where \(\hat{S}(t, 0)\) is the unitary evolution operator
and $|\Phi(0)\rangle$ is an initial state of the system, is described in the Dyson expansion form with time ordering operator $T$. On the other hand the unitary evolution operator $\hat{S}(t, 0)$ can be also presented in the other form: so-called Magnus expansion, which seems to be more preferable for some specific systems than the standard Dyson expansion (see, for example [110, 128, 129]). In this representation the time-evolution operator is written as an ordinary exponential operator instead of chronological exponential operator

$$\hat{S}(t, 0) = T \exp \left( -i \int_{0}^{t} \hat{h}(t') dt' \right) = \exp(\hat{\Omega}(t)), \quad (4.36)$$

where $\hat{\Omega}(t = 0) = 0$ and a series Magnus expansion for the operator in the exponent takes place

$$\hat{\Omega}(t) = \sum_{k=1}^{\infty} \hat{\Omega}_{k}(t). \quad (4.37)$$

For instance, the three terms of the series are given as

$$\hat{\Omega}_{1}(t) = -i \int_{0}^{t} \hat{h}(t') dt',$$

$$\hat{\Omega}_{2}(t) = -\frac{1}{2} \int_{0}^{t} dt' \int_{0}^{t'} dt'' [\hat{h}(t'), \hat{h}(t'')], \quad (4.38)$$

$$\hat{\Omega}_{3}(t) = \frac{i}{6} \int_{0}^{t} dt_{1} \int_{0}^{t_{1}} dt_{2} \int_{0}^{t_{2}} dt_{3} \left( [\hat{h}(t_{1}), [\hat{h}(t_{2}), \hat{h}(t_{3})]]ight)$$

$$+ [\hat{h}(t_{3}), [\hat{h}(t_{2}), \hat{h}(t_{1})]] \right).$$

Thus, the Magnus approach provides the approximate exponential representations of the time-evolution operator of the system which for two level atom can be represented in the simple form for all terms of the series expansion (4.37). Indeed, the all terms of Magnus expansion with the Hamiltonian (4.34) are only expressed through the Pauli matrices $\hat{\sigma}_{x}, \hat{\sigma}_{x}, \hat{\sigma}_{x}$ and hence the
following representations of time-evolution operator can be derived

\[
\hat{S}(t, 0) = \exp(\hat{\Omega}(t)) = \exp \left(-i\vec{G}(t)\hat{\sigma}\right)
= \hat{I} \cos(|\vec{G}(t)|) - i(\vec{\rho}(t)\hat{\sigma}) \sin(|\vec{G}(t)|),
\]

(4.39)

where the vector \( \vec{\rho}(t) = \vec{G}(t)/|\vec{G}(t)| \). In this way, the problem is reduced to calculation of the function \(|\vec{G}(t)|\) and the vector \( \vec{\rho} \) as the series of Magnus expansion. Up to the second-order Magnus expansion we obtain from Eqs. (4.38)

\[
\begin{align*}
G_x(t) &= \Delta \int_0^t dt' \cos(2\Lambda(t')), \\
G_y(t) &= \Delta \int_0^t dt' \sin(2\Lambda(t')), \\
G_z(t) &= \Delta^2 \int_0^t dt' \int_0^{t'} dt'' \sin \left[2(\Lambda(t'') - \Lambda(t'))\right],
\end{align*}
\]

(4.40)

and \( G(t) = |\vec{G}(t)| = \sqrt{G_x^2(t) + G_y^2(t) + G_z^2(t)} \). Thus, the vector state reads as

\[
|\Phi(t)\rangle = \exp \left(-i\vec{G}(t)\hat{\sigma}\right) |\Phi(0)\rangle
\]

(4.41)

and hence

\[
|\Psi(t)\rangle = \hat{U}(t) \exp \left(-i\vec{G}(t)\hat{\sigma}\right) |\Psi(0)\rangle.
\]

(4.42)

On the whole for the probability amplitudes of qubit states

\[
|\Psi(t)\rangle = C_1(t)|1\rangle + C_2(t)|2\rangle
\]

(4.43)
we obtain

\begin{align}
C_1(t) &= e^{i\Lambda(t)}[\cos(G(t)) - i\rho_z \sin(G(t))]C_1(0) \\
&\quad + e^{i\Lambda(t)}(-i\rho_x - \rho_y)\sin(G(t))C_2(0), \\
C_2(t) &= e^{-i\Lambda(t)}(-i\rho_x + \rho_y)\sin(G(t))C_1(0) \\
&\quad + e^{-i\Lambda(t)}[\cos(G(t)) + i\rho_z \sin(G(t))]C_2(0).
\end{align}

(4.44a)

(4.44b)

For the system that initially was in the ground state \( |1\rangle \), \( C_1(0) = 1 \), \( C_2(0) = 0 \), the population distribution \( P_2(t) \) of the state \( |2\rangle \) is

\[
P_2(t) = (1 - \rho_z^2) \sin^2(G(t)).
\]

(4.45)

We notice that this result is obtained in the most general form for arbitrary time-dependent field amplitude \( A(t) \) and for a wide range of external field frequencies. In the general case, however, there is not compact expressions for the components of the function \( G(t) \) that are given as the Magnus series expansions. The first term of this expansion (4.37) coincides exactly with the simple exponent solution of time-evolution operator with the Hamiltonian \( \hat{h}(t) \). The every other \( n \)-th term \( \hat{\Omega}_n(t) \) contains multiple integral of combinations of \( n - 1 \) commutators containing \( n \) Hamiltonian operators \( \hat{h}(t) \). Nevertheless, the result (4.45) has some advantages from a computational point of view. Indeed, it is easy to observe that in Furry representation each \( n \)-th term \( G(t) \) of the Magnus expansion is proportional to \( \Delta^n \) thus an iteration procedure can be realized with the small parameter \( \Delta/\varepsilon_0 < 1 \). Particularly, it is easy to get the simplest expressions for the population of qubit in RWA describing interaction of two-level system and a single electromagnetic mode for the case when the frequency of the mode is near resonance with the qubit transition frequency and the coupling constant of interaction is weak (for atomic systems, see [86–88]).
4.2.2 Multiphoton resonant excitations

In this subsection the case of multiphoton resonant interaction with pulses is considered. The condition of resonance is formulated for the frequency $\omega$ and the electronic energy difference as $\varepsilon_0 - N\omega = \Delta_N \ll \varepsilon_0$. Under this condition the oscillating terms in time are vanished in the expressions of Eqs. (4.40).

Single-pulse resonant excitation

At first, let’s consider Eqs. (4.30), (4.35) for a single pulse in the following approximation form

$$\Lambda(t) = \frac{1}{2} \varepsilon_0 t + \int_0^t A(t') \cos(\omega t' + \theta) dt' = \frac{1}{2} \varepsilon_0 t + A(t) \frac{\sin(\omega t + \theta)}{\omega}$$

(4.46)

for the case of adiabatic pulses for with duration $T_D \gg \frac{2\pi}{\omega}$. Particularly for pulses with Gaussian envelope

$$A(t) = A_0 e^{-\frac{(t-t_0)^2}{T^2}}$$

(4.47)

this condition is equivalent to $T \gg \frac{2\pi}{\omega}$, where $T \sqrt{2}$ is the Gaussian RMS width.

While calculating Eqs. (4.40) with the approximated formula (4.46) we also use the well known formulas for the Bessel functions. As an intermediate result the following formula can be obtained

$$e^{2i\Lambda(t)} = e^{i(\varepsilon_0 t + 2A \sin(\omega t + \theta))} = e^{i\varepsilon_0 t} \sum_n J_n \left(\frac{2A(t)}{\omega}\right) e^{in(\omega t + \theta)},$$

(4.48)
where \( J_n(x) \) is \( n \)-th order Bessel function of the first kind. For the case of exact \( N \)-th order resonance, \( \varepsilon_0 = N\omega \), neglecting fast oscillating terms we get

\[
e^{2i\Lambda(t)} = J_{-N} \left( \frac{2A(t)}{\omega} \right) e^{-iN\theta} = (-1)^N J_N \left( \frac{2A(t)}{\omega} \right) e^{-iN\theta}. \tag{4.49}
\]

In this approximation the Hamiltonian (4.34) transforms to

\[
\hat{h}(t) = (-1)^N \Delta J_N \left( \frac{2A(t)}{\omega} \right) \left[ \hat{\sigma}_+ e^{iN\theta} + \hat{\sigma}_- e^{-iN\theta} \right]. \tag{4.50}
\]

Thus, the components of the vector \( \vec{G}(t) \) up to second-order of Magnus expansion are calculated as

\[
G_x(t) \approx (-1)^N \Delta \cos(N\theta) \int_0^t J_N \left( \frac{2A(t')}{\omega} \right) dt',
\]

\[
G_y(t) \approx (-1)^N \Delta \sin(N\theta) \int_0^t J_N \left( \frac{2A(t')}{\omega} \right) dt', \tag{4.51}
\]

\[
G_z(t) \approx 0.
\]

As we see, these Eqs. (4.51) don’t involve the terms of second-order as has been noted in Introduction. The component \( G_z(t) \) comes from the second-order Magnus term and hence is equal to zero in RWA approximation. It is easy to explain this truncation of Magnus series in RWA from the general point of view. Really, the high-order terms of Magnus expansion \( \hat{\Omega}_2(t), \hat{\Omega}_3(t), \ldots \) involve commutators of Hamiltonian operators \([\hat{h}(t_1), \hat{h}(t_2)], [\hat{h}(t_1), [\hat{h}(t_2), \hat{h}(t_3)]], \ldots \) that are equal to zero for the case of Hamiltonian in RWA (see, formula (4.50)). Thus, in RWA the operator of time-evolution is determined only by lowest exponential term involved \( \hat{h}(t) \).

On the whole for the population of excited state we obtain the following result

\[
P_2(t) = \sin^2 \left( \Delta \int_0^t J_N \left( \frac{2A(t')}{\omega} \right) dt' \right). \tag{4.52}
\]
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Figure 4.3: The population probability of excited state $P_2(t)$ for the case of single pulse excitation in the first-order resonance as a function of time. The parameters are: $\Delta/\varepsilon_0 = 0.3, T\varepsilon_0 = 10, \omega/\varepsilon_0 = 1$. Solid line (a), $A_0/\varepsilon_0 = 0.19$; dashed line (b), $A_0/\varepsilon_0 = 0.12$; dotdashed line (c), $A_0/\varepsilon_0 = 0.25$.

The results corresponding to the case of first-order resonance with initial conditions $C_1(0) = 1, C_2(0) = 0$ are depicted in Fig. 4.3 plotted using the formula (4.52). Here and in subsequent calculations in this article for the chosen pulse envelope (4.47) the calculation of the integrals of $J_N (2A(t)/\omega)$ is done numerically. As we see, the probability allows for a region of high transition probability with increasing the amplitude of single pulse. In the case of second-order resonance the transition probability $P_2(t)$ is small in contrast to the first-order resonance configuration. As will be demonstrated below the complete population inversion takes place for both resonant configurations in cases of multi-pulse excitations.

Phase-dependent two-pulse resonant excitation

In this subsection the dynamic of qubit in the result of two-pulse excitation is considered. We assume that the external field consists of two pulses with the same frequencies, but with different phases and amplitudes shifted on time-interval, $A(t) = A_1(t) + A_2(t)$, where $A_i(t) = A_0 e^{-(t-t_i)^2/T^2}, i = 1, 2$. The exponential function in this case reads as

$$\Lambda(t) = 1/2 \varepsilon_0 t + A_1(t) \frac{\sin(\omega t + \theta_1)}{\omega} + A_2(t) \frac{\sin(\omega t + \theta_2)}{\omega} \quad (4.53)$$
and the following expression can be obtained
\[
e^{2i\Lambda(t)} = \sum_{n_1,n_2} J_{n_1} \left( \frac{2A_1(t)}{\omega} \right) J_{n_2} \left( \frac{2A_2(t)}{\omega} \right) e^{i\varepsilon_0 + (n_1+n_2)\omega t} e^{in_1\theta_1} e^{in_2\theta_2}. \tag{4.54}
\]

We assume multiphoton resonant interaction if \( \varepsilon_0 - N\omega = 0 \), where \( n_1 + n_2 = -N \). Here, \( N \)-th order resonant interaction is realized by \( n_1 \) and \( n_2 \) photons correspondingly from first and second pulses. In the case when two pulses are far from each other and there is not simultaneous interaction the resonance condition can be written as \( n_j + N = 0 \) for every \( j \)-th pulse.

In RWA we reach
\[
e^{2i\Lambda(t)} = (-1)^N e^{-iN\theta_2} J_N(w(t)) \left( \frac{Z - ze^{-i\phi}}{Z - ze^{i\phi}} \right)^{\frac{N}{2}}, \tag{4.55}
\]
where \( w(t) = (Z^2 + z^2 - 2zZ \cos(\phi))^{1/2} \), \( z = \frac{2A_1(t)}{\omega} \), \( Z = \frac{2A_2(t)}{\omega} \), \( \phi = \theta_1 - \theta_2 + \pi \).

This result can be applied for time evolution that involves independent interactions with pulses, i.e. there is time interval between two interactions subjected by pulses, separated by interaction-free interval. Each of the independent interactions are described by a slowly varying RWA Hamiltonian (4.50). Nevertheless, we assume that pulses depend on the phases of the field envelopes. Thus, in this regime the complete expression for exponential function for an arbitrary moment in time reads as
\[
e^{2i\Lambda(t)} = (-1)^N \left[ J_N \left( \frac{2A_1(t)}{\omega} \right) e^{-iN\theta_1} + J_N \left( \frac{2A_2(t)}{\omega} \right) e^{-iN\theta_2} \right]. \tag{4.56}
\]

The general expression for the case of two pulses the population of excited state is calculated from (4.40), (4.45) and (4.55) as
\[
P_2(t) = \sin^2 \left( \Delta \sqrt{v(t)} \right), \tag{4.57}
\]
where
\[ v(t) = j_1^2(t) + j_2^2(t) + 2\cos(N(\theta_2 - \theta_1))j_1(t)j_2(t) \]  
(4.58)
and
\[ j_k(t) = \int_0^t J_N \left( \frac{2A_k(t')}{\omega} \right) dt'. \]  
(4.59)

This expression takes place for the general case of multi-order resonant interactions in term of the factor \( N \). As we see, in this configuration the controlling phase factor involved in the probability is the phase difference between consecutive pulses.

Now we focus on investigation of phase-dependent effects for one-photon resonant configuration considering pulses with Gaussian envelopes (4.47) that are shifted on the time-interval \( \tau \) between centers of the Gaussians. The results based on formula (4.57) for the population of excited state for two cases of phase delay between pulses are depicted in the Fig. 4.4 and Fig. 4.5. As we can see, high-level of the population \( P_2(t) \) is realized for two-pulse excitation depending on the duration of pulses \( T \) and time-interval between them \( \tau \). In Fig. 4.4, if the envelope phase difference between two pulses is zero, \( \Delta \theta = \theta_2 - \theta_1 = 0 \), we observe that time-evolution of the population strongly depends on the amplitude of pulse field and the population inversion \( P_2(t) - P_1(t) = 2P_2(t) - 1 \) is realized for definite time-intervals. If the phase delay is changed to \( \pi \), i.e. the difference between phases \( \Delta \theta = \theta_2 - \theta_1 = \pi \), the effect of the second pulse reverses the excitation process. The typical results are depicted in Fig. 4.5 for three values of the field amplitude. As we see, the curves indicate controllable excitation of qubit state by the first pulse and its decay produced by the second pulse. We conclude that these results depicted in Figs. 4.4 and Figs. 4.5 containing the interference effects between pulses are essentially differ from the analogous results for one-pulse regime (see, Fig. 4.3). The pulses are independent, if time-interval between them is sufficiently larger than \( T \).

It is easy to realize from the formula (4.56) that dependence on the phase
shift in probabilities varies with the order of the resonance due to the factor $\cos(N(\theta_2 - \theta_1))$. The order of the resonance reduces the required phase difference for observed effects. For example, in the case of second-order resonance the effect of probability raising and reducing analogous to the result shown on Fig. 4.5 takes place for $\Delta \theta = \pi/2$, instead of $\pi$, as it is for the case of first order resonance.
Qubit excitation beyond RWA

In this subsection we shortly discuss above introduced approach in the regime beyond RWA, i.e. for the Rabi pulsed model in the regimes where the RWA can not be applied. In this way, we consider single-qubit off-resonance interaction with strong external Gaussian pulse. We calculate numerically qubit populations due to non-resonant excitation using the general formulas (4.44) and (4.45) with high-order Magnus terms. As it can be seen from Eqs. (4.40), the first-order Magnus terms (proportional to \( \Delta \)) contain \( G_x(t) \) and \( G_y(t) \) components of the vector \( \vec{G}(t) \), while the second-order term (proportional to \( \Delta^2 \)) involves only \( z \) component, \( G_z(t) \). On the whole, the Magnus expansion up to the second-order involve all components of the Bloch vector in Eq.(4.45) and third-order and high-order terms can only modify these components. As can be seen from the calculations the small parameter of this expansion is the ratio \( \Delta/\varepsilon_0 < 1 \). Thus, below we study numerically the temporal evolution of the qubit state vector \( (4.43), (4.44) \) for small values of \( \Delta/\varepsilon_0 \) up to the second-order Magnus terms.

Note, that in calculations based on the formulas (4.40) the terms depending on \( \omega + \varepsilon_0 \) oscillate very rapidly and have been neglected in RWA (see, subsection A). In this subsection, we will explore the effects of these counter-rotating terms including oscillation terms beyond RWA with calculations based on the formulas (4.40). The typical results based on (4.40) and (4.45) are depicted in Fig.4.6 for three cases of non-resonance interaction with the frequencies of the external field: \( \omega/\varepsilon_0 = 0.5, \omega/\varepsilon_0 = 1.1 \) and \( \omega/\varepsilon_0 = 1.5 \). As we see, in this beyond RWA approach the population displays time-dependent oscillations due to involvement of counter-rotating terms in qubit dynamics in contrast to analogous results for the resonant case depicted in Fig.4.3. The other peculiarity with respect to the resonance case is that population of qubit decreases in time during the pulse. This effect is stipulated by the negative term \( \rho^2_z \) in Eq. (4.45) appearing in the second-order Magnus expansion. The Fig.4.6 also shows an asymmetry in non-resonant excitation of
Figure 4.6: The population probability of excited state $P_2(t)$ for the case of single pulse excitation beyond RWA as a function of time for initial conditions $C_1(0) = 1, C_2(0) = 0$ and the parameters: $\Delta/\varepsilon_0 = 0.45, A_0/\varepsilon_0 = 0.4, T\varepsilon_0 = 3.5$. Solid line (a), $\omega/\varepsilon_0 = 0.5$; dashed line (b), $\omega/\varepsilon_0 = 1.5$; dotdashed line (c), $\omega/\varepsilon_0 = 1.1$.

qubit state connected with values of the frequency relatively to the energy difference between ground and excited states of the qubit. Indeed, the population for the case of $\omega/\varepsilon_0 = 1.5$ exceeds analogous result for $\omega/\varepsilon_0 = 0.5$. It should be mentioned that calculation of the third-order Magnus term (see, Eqs. (4.38)) for used parameters only slightly changes the obtained results. Indeed, as it has been mentioned earlier the $n$-th term $G(t)$ of Magnus expansion is proportional to $\Delta^n$, thus the third term is proportional to $\Delta^3$ making it smaller than previous term of the expansion if $\Delta < 1$. On the other hand, as it can be concluded from the Eqs. (4.34) and (4.38), the third term $\Omega_3(t)$ is a superposition of only $\hat{\sigma}_x$ and $\hat{\sigma}_y$ operators and not $\hat{\sigma}_z$, thus it contributes only to $G_x(t)$ and $G_y(t)$, but not to $G_z(t)$. Which means that in Eq. (4.45) only sinusoidal part $\sin^2(G(t))$ is affected by consideration of the third term, while $1 - \rho_z^2(t)$ is the same for calculations to both second and third terms of Magnus expansion.
4.2.3 Excitation by identical pulses: formation of quasienergetic states

Another interesting configuration that can be realized involves the train consisting of identical pulse envelopes with the same phases. The trains consisting of big numbers of pulses approximately display periodicity in time that can be exploited for realization of new regimes of qubit dynamics.

For a sequence of pulses with identical phases $\theta_k = \theta = 0$ from Eqs. (4.51) we have $G_y(t) = 0$, and the function $G_x(t) = G(t)$ can be written as

$$G(t) = \Delta \sum_{k=1}^{M} \int_{0}^{t} J_N \left( \frac{2A_k(t')}{\omega} \right) dt' = \Delta \int_{0}^{t} J_N \left( \frac{2A(t')}{\omega} \right) dt',$$

where

$$A(t) = \sum_{k=1}^{M} A_k(t).$$

This formula is obtained in approximation of not overlapping envelopes, where interactions conditioned by pulses are separated by interaction-free time intervals. For train of Gaussian pulses we can consider the amplitude as the periodic function

$$A(t) = A_0 \sum_{k=-\infty}^{\infty} e^{-\frac{(t-k\tau)^2}{T^2}}$$

with the period equals to time intervals between pulses $\tau$. Thus, calculation of the function $G(t)$ contains integration of a periodic function $J_N \left( \frac{2A(t')}{\omega} \right)$. It is easy to represent the function $G(t)$ as the sum of a term with linear dependence on time and a periodic on time term

$$G(t) = \Delta \gamma_N t + \phi_N(t).$$
Here, $\phi_N(t)$ is a periodic function with period $\tau$ defined in $t \in [t_0, t_0 + \tau]$ as

$$\phi_N(t) = \Delta \int_{t_0}^{t} \left( J_N \left( \frac{2A(t')}{\omega} \right) - \gamma_N \right) dt'$$  \hspace{1cm} (4.64)$$

and

$$\gamma_N = \frac{1}{\tau} \int_{t_0}^{t_0 + \tau} J_N \left( \frac{2A(t')}{\omega} \right) dt'$$  \hspace{1cm} (4.65)$$
is the Bessel function averaged on the period. It is easy to realize that the last expression can be written through the integral of Bessel function for just one of the identical pulse envelopes as

$$\gamma_N = \frac{1}{\tau} \int_{-\infty}^{\infty} J_N \left( \frac{2A_k(t)}{\omega} \right) dt.$$

The population of the excited states (if the system was initially in the lower state, $C_1(0) = 1, C_2(0) = 0$) as a function of time is then given by

$$P_2(t) = \sin^2 \left( \Delta \gamma_N t + \phi_N(t) \right).$$  \hspace{1cm} (4.67)$$

according to Eq.(4.52). It should be mentioned that the formula (4.63) allows to introduce the quasienergetic states and quasienergies for the qubit interacting with time-periodic sequence of pulses. Indeed, it is easy to check that the solution of Eq. (4.41) for this configuration can be expressed in the adiabatic basis

$$|\pm\rangle = \frac{1}{\sqrt{2}} (|1\rangle \pm |2\rangle)$$  \hspace{1cm} (4.68)$$
as

$$|\Phi_{\pm}^N\rangle = e^{\mp iG(t)}|\pm\rangle = e^{\mp i(E_N t + \phi_N(t))}|\pm\rangle$$  \hspace{1cm} (4.69)$$
and then in the form of QES

$$|\Phi_{\pm}^N\rangle = e^{-iE_N^\pm t}U_{N}^\pm(t)|\pm\rangle,$$

(4.70)
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with the quasienergies

\[ E_N^+ = -E_N^- = E_N = \frac{\Delta}{\tau} \int_{-\infty}^{\infty} J_N \left( \frac{2A_k(t)}{\omega} \right) \, dt. \]  

(4.71)

Here

\[ U_N^\pm(t) = e^{\mp i\phi_N(t)} \]  

(4.72)

is periodic on time in accordance with denotation of quasienergetic states

\[ U_N^\pm(t + \tau) = U_N^\pm(t). \]  

(4.73)

It is important also to rewrite quasienergies with the input of qubit energy levels. In this way, taking into account the relation

\[ |\Psi_\pm_N(t)\rangle = e^{i\Lambda(t)\hat{\sigma}_z}|\Phi_\pm_N(t)\rangle, \]  

(4.74)

we get the states of driven qubit in the following form

\[
|\Psi_N^+(t)\rangle = e^{-i(\varepsilon_1 + E_N^+)t}U_N^+(t) \left[ e^{-i \sum_k A_k(t) \cos(\omega t + \theta_k)} |1\rangle + e^{i\varepsilon_0 t} e^{i \sum_k A_k(t) \cos(\omega t + \theta_k)} |2\rangle \right], \\
|\Psi_N^-(t)\rangle = e^{-i(\varepsilon_2 + E_N^-)t}U_N^-(t) \left[ e^{i\varepsilon_0 t} e^{-i \sum_k A_k(t) \cos(\omega t + \theta_k)} |1\rangle - e^{i \sum_k A_k(t) \cos(\omega t + \theta_k)} |2\rangle \right].
\]  

(4.75)

In these wave functions we added the phase factors involving the sum of qubit energetic levels, that is zero for the case of the truncated Hamiltonian (4.29), in which the half of the sum of qubit energetic levels has been omitted.

From the last equations the relation between real energies and quasienergies can be written as

\[ E_{1,N} = \varepsilon_1 + E_N^+ = \varepsilon_1 + E_N, \]  

(4.76a)

\[ E_{2,N} = \varepsilon_2 + E_N^- = \varepsilon_2 - E_N. \]  

(4.76b)
Figure 4.7: The typical aperiodic behaviour of the population of excited state $P_2(t)$ of the qubit interacting with a train of identical pulses as a function of time. The parameters: $\Delta/\varepsilon_0 = 0.3$, $T\varepsilon_0 = 10$, $\omega/\varepsilon_0 = 1$ and $\tau/T = 4$. Solid line (a) describes resonance of the first order, with $A_0/\varepsilon_0 = 0.19$; dashed line (b) describes resonance of the second order, with $A_0/\varepsilon_0 = 0.5$.

Note, that the sum of two quasienergies obeys the relation $E_{N}^{+} + E_{N}^{-} = 0$, while $E_{1,N} + E_{2,N} = \varepsilon_1 + \varepsilon_2$. This result is in accordance with the exact result taking place for a two-level atom in a periodic field [69].

Now we analyse the time-evolution of the probability of excitation (4.67). The typical results based on formula (4.67) are depicted in Fig.4.7 for two cases of first-order and second-order resonances. As we see the populations for both resonance configurations show complete population inversion for definite time-intervals. Such result is cardinaly different from the case of single-pulse excitation, where the population $P_2(t)$ for the second-order resonance is small. The function $G(t) = \Delta \gamma_N t + \phi_N(t)$ is an increasing function in time but it grows also periodically due to its “linear+periodic” structure. Therefore, the dynamics of populations Eq. (4.67) seems to be aperiodic in time as it is demonstrated in Fig. 4.7.

An interesting regime of Rabi oscillations can be formed if the following condition for the quasienergy takes place

$$\Delta \gamma_N \tau = E_N \tau = \pi m$$

(4.77)
FIGURE 4.8: Regular periodic behaviour of the population of excited state $P_2(t)$ of the qubit interacting with a train of identical pulses in time under condition $E_N \tau = \pi$. Solid line (a) describes resonance of the first order, $\Delta/\varepsilon_0 = 0.3$, $T_0 = 20$, $\tau/T = 3$ and $A_0/\varepsilon_0 = 0.315$; dashed line (b) describes resonance of the second order, $\Delta/\varepsilon_0 = 0.4$, $T_0 = 65$, $\tau/T = 2.15$ and $A_0/\varepsilon_0 = 0.457$.

or in the following form

$$\Delta \int_{-\infty}^{\infty} J_N \left( \frac{2A_k(t)}{\omega} \right) dt = \pi m, \quad \text{(4.78)}$$

where $m = 1, 2, \ldots$. In this case we observe that $G(t + \tau) = \Delta \gamma_N t + \phi_N(t) + \pi m = G(t) + \pi m$ and hence the transition probability $P_2(t) = \sin^2(G(t))$ shows periodicity on time-intervals. Thus, for the definite values of quasienergy the transition probability displays regular, periodic dynamics. The typical results showing clear regular behaviour are depicted in Fig. 4.8.

In the end of this section we analyse the quasienergy as a function of the amplitude of pulses and their duration. Calculations on the base of Eq. (4.71) lead to large values of the quasienergy in comparison with the analogous results for two-level atomic system driven by monochromatic field. As we see in Fig. 4.9 and Fig. 4.10 the quasienergy increases almost linearly with increasing the amplitude of pulses as well as with the duration of pulses for both resonant configurations.
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Figure 4.9: Quasienergy $E_N$ depending on the pulse amplitude in units of $\varepsilon_0$. The parameters are: $\Delta/\varepsilon_0 = 0.3$, the pulse width $T\varepsilon_0 = 10$, $\tau/T = 4$, $\omega/\varepsilon_0 = 1$. Solid line (a) describes the first-order resonant configuration; dashed line (b) describes the second order resonance.

Figure 4.10: Quasienergy $E_N$ depending on pulse duration. The parameters are: $\Delta/\varepsilon_0 = 0.4$, $\omega/\varepsilon_0 = 1$, $\tau\varepsilon_0 = 40$, the amplitude $A_0/\varepsilon_0 = 0.38$. Solid line (a) describes the case of first-order resonance; dashed line (b) describes the case of second-order resonance.

4.3 System of two qubits in the Furry-Mugnus approach

Here, the generalization of the method to the case of interacting qubits is considered. The Hamiltonian of the system of many qubits can be written as

$$\hat{H}(t) = \hat{H}_0 - \sum_{a=1}^{N} \frac{\Delta^a}{2} \hat{\sigma}_x^{(a)}, \quad (4.79)$$
\[ 
\hat{H}_0 = -\sum_{a=1}^{N} g_a(t) \hat{\sigma}^{(a)}_z + \frac{1}{2} \sum_{a,b} J_{a,b} \hat{\sigma}^{(a)}_z \hat{\sigma}^{(b)}_z , 
\] 
(4.80)

where different qubits differ by \( a \) and \( b \) indices. The quantity

\[ 
g_a(t) = \frac{1}{2} \varepsilon_a + A_a(t) \cos(\omega_a t + \theta_a) 
\] 
(4.81)

describes the interaction of each of the qubits with the external field of frequency \( \omega_a \), amplitude \( A_a(t) \) and phase \( \theta_a \). The Hamiltonian \( \hat{H}_0 \) includes the amplitudes of tunnel transitions and the interaction term of qubits with a coupling constant \( J_{a,b} \).

In this case, the operator of the Furry representation has the form

\[ 
|\Psi(t)\rangle = \hat{R}(t)|\Phi(t)\rangle , 
\] 
(4.82)

\[ 
\hat{R}(t) = \exp \left[ i \sum_a \Lambda_a(t) \hat{\sigma}^{(a)}_z - \frac{it}{2} \sum_{a,b} J_{a,b} \hat{\sigma}^{(a)}_z \hat{\sigma}^{(b)}_z \right] , 
\] 
(4.83)

\[ 
\Lambda_a(t) = \int_0^t g_a(t') dt' , 
\] 
(4.84)

and the state vector in the Furry representation may be written in the Dyson form with the Hamiltonian (4.79), and in the form of Magnus

\[ 
|\Psi(t)\rangle = T \exp \left( -i \int_0^t \hat{h}_{mq}(t')dt' \right) = \exp(\hat{Q}_{mq}(t)) . 
\] 
(4.85)

Here, the Hamiltonian in the Furry representation is equal to

\[ 
\hat{h}_{mq}(t) = -\frac{1}{2} \hat{R}^{-1}(t) \left( \sum_a \Delta_a \hat{\sigma}^{(a)}_x \right) \hat{R}(t) 
\] 
(4.86)

and the operator \( \hat{Q}_{mq}(t) \) is given by the Magnus expansion similar to formulas (4.37) and (4.38). Note, the Hamiltonian (4.86) is written here in the
non-explicit form. The concrete results can be obtained for particular cases of several qubits.

Below are the results for two interacting qubits, when the Hamiltonian is written in the form

\[
\hat{H} = \hat{H}_0 - \frac{1}{2} \left( \Delta_1 \hat{\sigma}_x^{(1)} + \Delta_2 \hat{\sigma}_x^{(2)} \right),
\]

(4.87)

where

\[
\hat{H}_0 = -g_1(t) \hat{\sigma}_z^{(1)} - g_2(t) \hat{\sigma}_z^{(2)} + J \hat{\sigma}_z^{(1)} \hat{\sigma}_z^{(2)},
\]

(4.88)

and it is written in the Furry representation as

\[
\hat{h}_{mq}(t) = \hat{R}^{-1}(t) \left( \Delta_1 \hat{\sigma}_x^{(1)} + \Delta_2 \hat{\sigma}_x^{(2)} \right) \hat{R}(t),
\]

(4.89)

where

\[
\hat{R}(t) = \exp \left[ i \Lambda_1(t) \hat{\sigma}_z^{(1)} + i \Lambda_2(t) \hat{\sigma}_z^{(2)} - i J t \hat{\sigma}_z^{(1)} \hat{\sigma}_z^{(2)} \right].
\]

(4.90)

To perform specific transformations, we use the following well-known formulas with the Pauli matrices:

\[
e^{i \alpha \hat{\sigma}_x} \hat{\sigma}_x e^{-i \alpha \hat{\sigma}_x} = \cos(2\alpha) \hat{\sigma}_x - \sin(2\alpha) \hat{\sigma}_y,
\]

(4.91a)

\[
e^{i \alpha \hat{\sigma}_y} \hat{\sigma}_y e^{-i \alpha \hat{\sigma}_y} = \sin(2\alpha) \hat{\sigma}_x + \cos(2\alpha) \hat{\sigma}_y.
\]

(4.91b)

With the aid of formulas (4.91) one may obtain the transformation

\[
\hat{R}^{-1}(t) \hat{\sigma}_x^{(1)} \hat{R}(t) = \cos(2\Lambda_1(t)) \left( \hat{\sigma}_x^{(1)} \cos \left( 2 J t \hat{\sigma}_z^{(2)} \right) - \hat{\sigma}_y^{(1)} \sin \left( 2 J t \hat{\sigma}_z^{(2)} \right) \right)
\]

\[
- \sin(2\Lambda_1(t)) \left( \hat{\sigma}_x^{(1)} \sin \left( 2 J t \hat{\sigma}_z^{(2)} \right) + \hat{\sigma}_y^{(1)} \cos \left( 2 J t \hat{\sigma}_z^{(2)} \right) \right),
\]

(4.92)
which in matrix form looks like:

\[
\hat{R}^{-1}(t)\hat{\sigma}_x^{(1)}\hat{R}(t) = \begin{pmatrix}
0 & e^{-2iJt\Lambda_1(t)} & e^{2iJt\Lambda_2(t)} & e^{-2iJt\Lambda_2(t)} \\
e^{2iJt\Lambda_2(t)} & e^{2iJt\Lambda_1(t)} & 0 & 0 \\
e^{2i(Jt+\Lambda_1(t))} & 0 & 0 & 0 \\
e^{-2i(Jt-\Lambda_1(t))} & 0 & 0 & 0
\end{pmatrix}.
\] (4.93)

Similar expressions are also obtained for the second qubit. Combining these results, for the Hamiltonian of the system one can obtain in the Furry representation the following result

\[
\hat{h}_{mq}(t) = \Delta_1 \begin{pmatrix}
0 & 0 & e^{-2i(Jt+\Lambda_1(t))} & 0 \\
0 & 0 & 0 & e^{2i(Jt-\Lambda_1(t))} \\
e^{2i(Jt+\Lambda_1(t))} & 0 & 0 & 0 \\
e^{-2i(Jt-\Lambda_1(t))} & 0 & 0 & 0
\end{pmatrix} + \Delta_2 \begin{pmatrix}
0 & 0 & e^{-2i(Jt+\Lambda_2(t))} & 0 \\
0 & 0 & 0 & e^{2i(Jt-\Lambda_2(t))} \\
e^{2i(Jt+\Lambda_2(t))} & 0 & 0 & 0 \\
e^{-2i(Jt-\Lambda_2(t))} & 0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & \Delta_2 e^{-2i(Jt+\Lambda_2(t))} & \Delta_1 e^{-2i(Jt+\Lambda_1(t))} & 0 \\
\Delta_2 e^{2i(Jt+\Lambda_2(t))} & 0 & 0 & \Delta_1 e^{2i(Jt-\Lambda_1(t))} \\
\Delta_1 e^{2i(Jt+\Lambda_1(t))} & 0 & 0 & \Delta_2 e^{2i(Jt-\Lambda_2(t))} \\
0 & \Delta_1 e^{-2i(Jt-\Lambda_1(t))} & \Delta_2 e^{-2i(Jt-\Lambda_2(t))} & 0
\end{pmatrix}.
\] (4.94)

This Hamiltonian is represented as the sum of $4 \times 4$ matrix and operates in the space of four states of two qubits. The main advantage of such a Hamiltonian is that in the Furry representation it is proportional to the tunneling amplitudes, which have the meaning of interaction constants of system by
which one can formulate an iterative procedure. As to the interaction of qubits with the perturbation field, it is taken into account in all orders of interaction constants $g_a(t)$ and $g_b(t)$, so the results are also applicable for the case of intensive perturbation fields. It is obvious that the expressions (4.38) given for the Magnus expansion up to the terms of third-order are obtained by the simple replacing of $\hat{h}(t)$ by $\hat{h}_{mq}(t)$ in them. Each $m$-th order of the Magnus expansion includes the degrees of $m$ order of tunneling amplitudes.
Chapter 5

Conclusion

We have investigated atomic deflection in a bimodal cavity due to dispersive interactions of atoms with two standing waves in entangled state. It is well known that when atoms interact with a standing light wave in an optical cavity, information about the phase of the field is recorded in the position of the atom. On the other hand, atomic position information is encoded in the standing wave field and measurement of the phase is spatially localized atoms. We have studied these and the other problems for deflection of three level-atoms of $V$-type and $\Lambda$-type configurations interacting with co-propagating two cavity modes in entangled states. The conditional position distributions corresponding to joint measurement of the position of the atom and the relative phase of the fields has been calculated for two types of deflected atoms. Analogous studies have been also done in phase-space on the framework of the Wigner functions as well as for momentum distributions of deflected atoms. We have investigated the deflected patterns of the Gaussian atomic wave packet at the position of best localization. The concrete calculations of atomic distributions have been perform for atoms filtered by a slit of width $\Delta x = 0.1 \lambda \div 0.3 \lambda$ placed at a node of one of the modes. We have demonstrated that the results for the case of interaction of $V$-type atoms with entangled two standing waves are essentially different from the usual one for the atomic deflection on two statistically independent standing waves. We hope that these results can be serving as a guide for further studies of applications and testing of entanglement. It has been also demonstrated that the
modes entanglement is weakly displayed in deflection of Λ-type atoms due to coherent Raman processes. However, this system is best for investigation of deflection atoms in coherent superposition states. In this way, it has been demonstrated that the deflection pattern of Λ-type atoms are being initially in superposition of lower states is essentially differ from the analogous pattern corresponding to deflection of atoms in one of the lower states. We also have addressed the problem of the passage from the near field pattern to the far field pattern considering the available for experiments dates.

By calculating the concrete conditional spatial distributions of Λ-structured atoms after passing two crossed standing light waves we have shown that two-dimensional patterns of deflected atoms contain important information concerning the low-level atomic superposition states as well as reflect the efficiency of the two-photon resonant Raman process. Thus, we have developed an approach for testing and visualization of superposition states as well as for probing the Raman resonance in a new not spectroscopic manner. In more details, we have studied how various initial atomic superposition states $a|1⟩ + b|2⟩$ in the Gaussian atomic beam are visualized in both 2D and 3D distributions of the joint probabilities. This analyses has been done for both near- and far-field diffraction zones as well as in the momentum space. Considering two regimes of the interaction of Λ-type atoms with two-mode field we have clearly demonstrated on the spatial patterns the peculiarities of coherent Raman processes in comparison with corresponding one-photon processes from the point of view of atomic optics.

We have analyzed dynamics of a superconducting qubit interacting with an electromagnetic wave with time-modulated amplitude (or a bichromatic field) for two basic configurations that involve time-dependent components along $z$ or $x$ axes. In the case of the $z$ configuration, the external bichromatic field drives the qubit’s energetic levels, while in the case of $x$ configuration (describing also the standard problems of a two-level atom in a bichromatic field) the coupling with the bichromatic field leads to the transition dipole
moment between two states of atoms. We have calculated quasienergetic states and quasienergies of the composite system "superconducting qubit plus time-modulated field" in an adiabatic basis of the system analyzing the quasienergies numerically for arbitrary intensities of the external field as well as analytically in detail for the regime of weak driving. Considering the dependence of quasienergies from the intensity parameter $A/\omega_0$ we have shown oscillation-type behavior of quasienergies for the case of a strong bichromatic field. In this way, we demonstrate the drastic difference between QESs of two schemes. Particularly, for the standard two-level model in a bichromatic field (the $x$ configuration) the quasienergies are equal to zero for all ranges of the parameters that are displayed in the spectral line of RF and Autler-Townes splitting. In contrast to this case, the QES for the scheme involving time-dependent $z$-axis coupling has a more complicated structure. On the whole, the spectral lines of QES transitions contain also field-dependent Stark shifts due to the input of the quasienergies.

We have considered time dependence of the occupation probabilities of qubit states and Rabi physics for both first- and second-order resonance regimes, when the central frequency $\omega_0$ and the electronic energy difference obey rules $\varepsilon_0 = N\omega_0$, where $N$ is the order of the resonance. Considering Rabi oscillations between qubit states we have shown that these oscillations are aperiodic in time due to effects of time-dependent modulation. Nevertheless, further, we have demonstrated new regimes in which dynamics of populations becomes periodically regular. These regimes can be realized if the ratio of quasienergy to the detuning is positive integer $E_N/\delta = r$ for an arbitrary order of resonances. Together with the recent advancements in the engineering of various schemes of superconducting qubits, these results seem to be important for further studies of quantum phenomena in this area.

A general approach has been formulated based on the Magnus quantum electrodynamical formalism in Furry picture for analysing the Rabi model
with time-dependent interaction Hamiltonians along the $z$ axis. In this approach we have obtained the formal expressions (4.44) and (4.45) for the populations of a single-qubit interacting with time-dependent fields of arbitrary amplitudes and wide range of frequencies. We have investigated the dynamics and manipulations of qubit due to multiphoton resonant interactions with external pulses for various regimes of state excitation including: single-pulse excitation, two-pulse excitation with phase shift between pulse envelopes being controlling parameter and for excitation with identical sequential pulses. Beside this, we have shown efficiency of this approach for interaction of qubits beyond RWA; the corresponding iteration procedure includes high-order Magnus terms for small perturbative parameter $\Delta/\varepsilon_0$.

Considering two resonant configurations, i.e. first- and second-order resonant configurations, we have shown that the complete population inversion can be achieved through tailoring the pulse train for both configurations at definite time intervals. Particular emphasis has been laid on interaction of qubit with train of identical pulses. We have shown that quasienergies and quasienergetic states of combined system "qubit + pulse train" are forming for all multiphoton resonant regimes. The quasienergy increases almost linearly with increasing the amplitude of pulses as well as with the duration of pulses. Moreover, we have demonstrated that for all multiphoton resonant regimes the state population exhibits time-dependent aperiodic oscillations, but also displays periodically regular oscillations for the definite values of the quasienergy.

- The deflection of atomic beams on two co-propagating standing waves excited in a bimodal cavity has been investigated. The novelty is that the standing waves are in an entangled, photon-number correlated state and interactions with three-level atoms of $V$-type and $\Lambda$-type configurations are considered. The splitting of atomic states, quantum interference effects and spatial atomic localization due to phase measurements
is investigated on the framework of both the conditional position distributions of atomic wave packets as well as the Wigner functions for atomic translation variables. It is demonstrated that these quantities for $V$-type atoms passing the cavity are essentially modified when entangled states of standing waves are used instead of statistically independent waves. The deflection patterns and the Wigner functions are also modified if atoms are initially prepared in a superposition of low-level states. It is best demonstrated for the $\Lambda$-type configuration, where transition through the Raman resonance is realized, and proposed for testing superposition states in a new non-spectroscopic manner.

• Deflection of atoms in $\Lambda$-type configuration passing through two crossed standing light waves is proposed for probing and visualization of atomic superposition states. For this goal, we use both the large-dispersive and Raman-resonant regimes of atom-field interaction giving rise to a position-dependent phase shifts of fields and perform double simultaneous spatial measurements on an atom. In this way, it is demonstrated that the deflection spatial patterns of atoms in $\Lambda$-configuration passing through modes of standing waves are essentially modified if the atoms are initially prepared in a coherent superposition of its low levels states as well as when the superposition states are created during the process of deflection. The similar results take place for the joint momentum distribution of atoms. Further, considering both one-photon and two-photon excitation regimes of $\Lambda$-atoms we also illustrate that the two-dimensional patterns of defected atoms qualitatively reflects the efficiency of the Raman processes.

• The formation of two-dimensional spatial structures of atoms due to atomic diffraction on two crossed standing electromagnetic fields is discussed. This analysis proposed for $\Lambda$-type atomic configuration under dispersive atom-field interactions in quantum regime. Localization of
the position of atoms passing through standing light wave is initiated
by making a quadrature phase measurement on the light fields. We de-
velope the procedure for strong two-dimensional spatial localization of
atomic beam within the optical wavelength. Considering atomic spa-
tial localization in the presence of entanglement of two light beams vari-
ous two-dimensional patterns for Λ-type atoms are reported.

- The dynamics of a superconducting qubit and the phenomenon of mul-
tiorder Rabi oscillations in the presence of a time-modulated external
field is analyzed. Such a field is also presented as a bichromatic field
consisting of two spectral components, which are symmetrically de-
tuned from the qubit resonance frequency. This approach leads to ob-
taining qualitative quantum effects beyond those for the case of mono-
chromatic excitation of qubits. We calculate Floquet states and QEs
of the composite system superconducting qubit plus time-modulated
field for various resonant regimes. We analyze the dependence of QEs
from the amplitude of an external field, demonstrating the zeros of dif-
fERENCE BETWEEN QEs. We show that, as a rule, populations of qubit
states exhibit aperiodic oscillations, but we demonstrate the specific
important regimes in which dynamics of populations becomes period-
ically regular.

- A study of qubit manipulations by a train of pulses in a systematic
approach based on the Magnus expansion and Furry representation
in quantum electrodynamics is presented. Furry-Magnus expansion
gives approximate exponential representation of time-evolution oper-
ator without Dyson time-ordering procedure for interactions with ex-
ternal fields of arbitrary intensity and allows us to obtain general ex-
pression for single-qubit populations as a function of time-dependent
Rabi frequency. The concrete calculations are performed for tunneling
quantum dynamics, for multiphoton resonance interactions of qubit as
well as for off-resonance excitations of qubit driven by Gaussian pulses beyond rotating wave approximation. In this way, the populations of qubit states are investigated for various operational regimes including: single-pulse excitation, two-pulse excitation with phase shift between pulse envelopes being controlling parameter and for excitation with sequential pulses. In the last case, we demonstrate the formation of quasienergetic states and quasienergies of qubit driven by train of identical pulses. In this case the transition probability of the qubit generally exhibiting aperiodic oscillations, becomes periodically regular for definite values of the quasienergy.

These results are presented in 9 publications listed below.

**Publications:**


Bibliography


